On the Existence of Competitive Equilibrium with Chores

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Abstract

We study the chore division problem in the classic Arrow-Debreu exchange setting, where a set of agents want to divide their divisible chores (bads) to minimize their disutilities (costs). We assume that agents have linear disutility functions. Like the setting with goods, a division based on competitive equilibrium is regarded as one of the best mechanisms for bads. Equilibrium existence for goods has been extensively studied, resulting in a simple, polynomial-time verifiable, necessary and sufficient condition. However, dividing bads has not received a similar extensive study even though it is as relevant as dividing goods in day-to-day life.

In this paper, we show that the problem of checking whether an equilibrium exists in chore division is NP-complete, which is in sharp contrast to the case of goods. Further, we derive a simple, polynomial-time verifiable, sufficient condition for existence. Our fixed-point formulation to show existence makes novel use of both Kakutani and Brouwer fixed-point theorems, the latter nested inside the former, to avoid the *undefined demand* issue specific to bads.

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1 Introduction

Fair division has developed into a fundamental branch in mathematical economics, computational social choice theory and computer science over the last several decades. In a classical fair division problem, the goal is to divide a set of items among agents in a *fair* and *efficient* manner. Such problems have been extensively studied when the items to be divided are all *goods*. The problem of dividing *chores* (items creating disutility) has not received a similar extensive investigation even though it is as relevant as dividing goods in day-to-day life; for instance division of daily household chores among tenants, teaching load among faculty, job shifts among workers, and so on. A division based on *competitive equilibrium* (CE) has emerged as one of the best mechanisms for this problem due to its remarkable fairness and efficiency guarantees [Var74, BMSY17].

In this paper, we consider the problem of computing a CE with *divisible* chores in the fundamental Arrow-Debreu *exchange* model. The exchange model is like a barter system, where each agent brings a set of chores that needs to be completed and exchanges them with others to optimize their (dis)utility. For example, a set of university students teaching each other in a group study, to optimize the time and effort required. At a larger scale, *timebanks*[1] are such reciprocal service exchange platforms which have around 30,000 to 40,000 users from the United States. In a timebank, individuals from a certain community give services to one another and earn *time credit*. Thereafter, each individual uses their time credit to receive services. CE provides a systematic way to do the exchange: it constitutes of prices (payment)[2] for chores and an allocation such that all chores are completely assigned and each agent gets her most preferred bundle (*optimal bundle*) subject to her budget constraint[3].

We assume that agents have linear disutility (cost) functions, i.e., the disutility of an agent is $\sum_j d_{ij} X_{ij}$, where d_{ij} is the disutility agent i gets from doing a unit amount of chore j, and X_{ij} indicates the amount of chore j that agent i does. Clearly, an agent can do a chore within a reasonable amount of time only if she has the skill set required for it. For example, a professor trained in computer science (CS) can teach a CS course in the upcoming semester, but may not have skill set to teach a course in music. This essentially boils down to not allocating certain chores to certain agents. In the case of goods, this is achieved by specifying *zero* utility values to some items, and its analogue for chores is specifying *infinite* disutility.

The existence of CE is well understood for goods. In particular, when agents have (quasi-)concave utility functions, Arrow and Debreu [AD54], and Mckenzie [McK54, McK59] had shown the existence of CE under some mild conditions. Both the theorems make use of Kakutani's fixed point formulations. When the utility functions are further restricted to be linear, there are well known convex programs that capture the competitive equilibrium in the exchange model [NP83, Jai07, DGV16]. Such convex programs have been instrumental in designing polynomial time algorithms for finding a competitive equilibrium when agents have linear utility functions [Jai07, Ye08].

Interestingly, CE with chores behaves significantly differently than CE with goods. Bogomolnaia et al. [BMSY17] considered the CEEI (CE with equal income) model, a special case of the exchange where every agent owns one unit of every chore, with finite and homogeneous disutilities. They gave an involved characterization of CE, and through this showed that with chores, the set of CE is non-convex and disconnected even when disutility functions are restricted to linear. While, in the case of linear CEEI model with goods, there are even simpler convex programs [EG59, CDG$^+$17] that capture CE.

In this paper, we analyze existence of CE with chores in the exchange model where agents may have infinite disutility for certain chores. Although infinite disutilities seem natural, they create more challenges. For example, we observe that CE in the CEEI model may not exist in contrast to guaranteed existence with finite disutilities [BMSY17]. Furthermore, it is NP-hard to determine the existence in both exchange and

[1] `https://timebanks.org/`

[2] Equivalent of time credit in time banks.

[3] Here the budget constraint of an agent is that she has to earn enough to pay for her initial set of chores.

CEEI. This is in sharp contrast to the goods case, where there is a polynomial time verifiable necessary and sufficient condition for existence of CE in the exchange setting [Gal76, DGV16]. Our NP-hardness result rules out the possibility of obtaining such conditions for chores case unless P=NP! Furthermore, we strengthen our NP-hardness result to hold for $11/12$-approximate CE.

The next best hope is to obtain weakest possible sufficient conditions that also capture interesting instances, leading to our main question: *Are there polynomial-time verifiable, natural, sufficient conditions that guarantee the existence of a CE with chores?*

Our result address the above question. First, we show the existence of a CE under two conditions. The first condition, known as *strong connectivity of the exchange graph*, is an artifact of the exchange model, and is required in the case of goods as well [Max97, VY11]. Intuitively, it ensures that no set of agents can consume only a strict subset of the chores they cumulatively own, as otherwise no prices can ensure demand equals supply. Our second condition depends on the disutility values. While this condition is specific to only bads, it is simple, polynomial-time verifiable, and unavoidable (see Example 2).

The proof of existence of a CE under these two conditions makes use of Kakutani's as well as Brouwer's fixed-point theorems, with the latter nested inside the former. The fixed point formulations for the goods case define a *correspondence* (or equivalently a *set valued function*) on the *simplex* domain of prices [AD54, SS75, Max97]. The correspondence maps each price vector to a set of price vectors in the simplex obtained by adjusting the price of each good depending on its *excess demand*.[4] Thereafter, by Kakutani's fixed point theorem the correspondence admits a fixed point, which is mapped to a CE by showing no excess demand at a fixed-point.

With chores, the simplex domain of prices pose the issue of *undefined optimal bundles* of the agents: If an agent owns chores that have positive prices but all the chores she can do (has finite disutility towards) have zero prices, then there is no way she can earn the money needed for her endowment, thereby making her optimal bundle undefined. We fix this issue by adding a set of linear constraints to our price domain, which ensures that if the total prices of the chores an agent is interested in is zero, then her total endowment money is also zero, implying that she does not need to earn anything and the *doing-no-chores* is an admissible optimal bundle. However, such a fix makes it harder to define an appropriate correspondence: be mindful that given a price vector, we need our correspondence to adjust prices depending on excess demand as before, but now map it back to a more involved domain (earlier it was a simplex). Additionally, it should satisfy the continuity-like property. It is unclear whether a correspondence with all the desired properties exist. This is where we use Brouwer's fixed point theorem to show the existence of such a correspondence. An overview of this technique can be found in Section 1.2.1.

1.1 Model and Notations

A chore division problem consists of a set of m divisible chores (bads), namely $B = \{b_1, \dots, b_m\}$, and a set of n agents $A = \{a_1, \dots, a_n\}$. Each agent a_i has $d(a_i, b_j) \in (0, \infty]$ disutility (pain) for doing unit amount of chore b_j.[5] Here, infinite disutility implies that the agent does not have required skill set to do the chore in a reasonable amount of time. If agent a_i is assigned bundle $X_i = \langle X_{i1}, \dots, X_{im}\rangle \in \mathbb{R}^m_{\geq 0}$ where X_{ij} is the amount of chore b_j she gets, then her total disutility is $d_i(X_i) = \sum_{j\in[m]} d(a_i, b_j) \cdot X_{ij}$. We study the problem under exchange model, where agent a_i brings $w(a_i, b_j)$ amount of chore b_j to be done (by herself or other agents).

Given prices $p = \langle p(b_1), p(b_2), \dots, p(b_m)\rangle \in \mathbb{R}^m_{\geq 0}$ for chores, where $p(b_j)$ denotes the payment for doing unit amount of chore b_j, agent a_i needs to earn $\sum_{j\in[m]} w(a_i, b_j) \cdot p(b_j)$ in order to pay to get her own

[4] At any given price, an agent can be content with several allocations, i.e., there are multiple optimal bundles at a given price. As different optimal bundles can lead to different excess demands for the goods, such a correspondence maps a price vector to several price vectors in the simplex.

[5] If $d(a_i, b_j)$ is zero, then chore b_j can be safely assigned to agent a_i and can be removed from the instance.

chores done. In this light, we define the feasible set of bundles $F_i(p)$ as those bundles with which an agent can earn her required money, i.e., $F_i(p) = \left\{X_i \in \mathbb{R}^m_{\geq 0} \mid \sum_{j\in[m]} X_{ij} \cdot p(b_j) \geq \sum_{j\in[m]} w(a_i, b_j) \cdot p(b_j)\right\}$. Clearly a_i would like to choose the feasible bundle that minimizes her disutility – this defines her *optimal bundle* (or optimal chore set).

$$OB_i(p) = \underset{X_i \in F_i(p)}{\arg\min}\ d_i(X_i). \tag{1}$$

It is easy to see that in her optimal bundle agent a_i gets assigned only those chores that minimizes her disutility per dollar earned and agent i earns money exactly equal to the total price of her endowments. Formally, if $X_i \in OB_i(p)$, then,

$$\forall j \in [m],\ \ X_{ij} > 0 \ \ \Rightarrow\ \ \tfrac{d(a_i,b_j)}{p(b_j)} \leq \tfrac{d(a_i,b_{j'})}{p(b_{j'})}\ \ \forall j' \in [m],$$

and

$$\sum_{j\in[m]} X_{ij} \cdot p(b_j) = \sum_{j\in[m]} w(a_i, b_j) \cdot p(b_j).$$

In the above ratios, to deal with zero prices and infinite disutilities we assume $\infty/a > b/0$ for any $a, b \in [0, \infty)$. Clearly, an optimal bundle of an agent contains only those chores for which she has finite disutility.

Price vector p is said to be at a *Competitive Equilibrium (CE)* if all chores are completely assigned when every agent gets one of her optimal bundles, *i.e.,* $X_i \in OB_i(p)$ and $\sum_{i\in[n]} X_{ij} = \sum_{i\in[n]} w(a_i, b_j),\ \forall j \in [m]$. It is without loss of generality to assume that each chore is available in one unit total, i.e. for each $b_j \in B$, $\sum_{i\in[n]} w(a_i, b_j) = 1$ (through appropriate scaling of the disutility values). We now formally describe our problem.

Definition 1 (Chore Division in the Exchange Model)**.** *Given a set of agents* $A = \{a_1, a_2, \ldots, a_n\}$*, chores* $B = \{b_1, b_2, \ldots, b_m\}$*, disutilities* $d(\cdot,\cdot)$ *and endowments* $w(\cdot,\cdot)$*, our goal is to find a price vector* $p = \langle p(b_1), p(b_2), \ldots, p(b_m)\rangle \in \mathbb{R}^m_{\geq 0}$ *and allocation* $X = \langle X_1, X_2, \ldots, X_n\rangle$*, such that*

- *Every agent gets their optimal bundle:* $X_i \in OB_i(p)$*.*
- *All chores are completely allocated:* $\sum_{i\in[n]} X_{ij} = \sum_{i\in[n]} w(a_i, b_j) = 1$*, for all* $b_j \in B$*.*

Observe that the equilibrium prices are scale invariant: if p is an equilibrium price vector then so is $\alpha \cdot p$ for any positive scalar α. Furthermore, at equilibrium $p(b_j) > 0$ for each chore j, otherwise no agent would be willing to do it. A CE $\langle p, X\rangle$ has many desirable properties like envy-freeness and Pareto optimality in the chore division with equal income [BMSY17]. Similarly, CE for the exchange model too satisfies Pareto optimality and *weighted* envy-freeness[6].

Fisher Model and CEEI. The Fisher model is a special case of exchange model, where instead of the endowment of chores, each agent a_i has a requirement of earning a fixed amount of money $e(a_i) \geq 0$, i.e., the only change is in the definition of the feasible set of chores that can be allocated to an agent at a given price vector p, $F_i(p) = \left\{X_i \in \mathbb{R}^m_{\geq 0} \mid \sum_{j\in[m]} X_{ij} \cdot p(b_j) \geq e(a_i)\right\}$. If $e(a_i) = 1$ for all $a_i \in A$ then resulting equilibrium is called *Competitive Equilibrium with Equal Income* (CEEI). Clearly, CEEI is a special case of Fisher. Observe that determining CE in the Fisher model, can be modeled as determining CE in the exchange model, by setting $w(a_i, b_j) = e(a_i)$ for each $a_i \in A$ and $b_j \in B$, while keeping the disutility values as is.

[6]Weight of an agent at given prices is the total monetary cost of the chores she brings. Naturally, higher the cost of her chores (more money she has to earn), larger is her share of disutility.

1.2 Overview of Our Results and Techniques

In this section we discuss the high-level ideas and techniques used to prove our main results. We first note that in general, a chore division instance may not admit a CE as demonstrated by the following example.

Example 1. *There are two agents a_1 and a_2, and two chores b_1 and b_2. We have $w(a_i, b_j) = 1$ for all $i, j \in [2]$, and $d(a_1, b_1) = d(a_2, b_1) = 1$, and $d(a_1, b_2) = \infty$ and $d(a_2, b_2) = 2$. Let $p(b_1)$ and $p(b_2)$ be the prices of the chores at a CE.*

Observe that since $d(a_1, b_2) = \infty$, a_1 earns her entire money of $w(a_1, b_1) \cdot p(b_1) + w(a_1, b_2) \cdot p(b_2)$ from b_1. Therefore, at a CE, the total price of the chore b_1 is at least the total money earned by a_1: $(w(a_1, b_1) + w(a_2, b_1)) \cdot p(b_1) \geq (w(a_1, b_1) \cdot p(b_1) + w(a_1, b_2) \cdot p(b_2))$. This implies that $2 \cdot p(b_1) \geq p(b_1) + p(b_2)$, further implying that $p(b_1) \geq p(b_2)$. In that case observe that the disutility to price ratio of b_2 is strictly less than that of b_1 for a_2: $d(a_2, b_1)/p(b_1) = 1/p(b_1) < 2/p(b_1) \leq 2/p(b_2) = d(a_2, b_2)/p(b_2)$. Thus, none of the agents are willing to do chore b_2, and therefore it remains unassigned, a contradiction.

It is well known that the a CE may not exist while dividing goods as well under the exchange model. And, there are *polynomial time checkable* necessary and sufficient conditions for the existence of CE. The next natural question is to obtain similar conditions for the chore division as well. However, in Section 3 we prove the following theorem.

Theorem 1. *Determining whether an instance of chore division in the Fisher model admits a CE is strongly NP-hard, even for the case of equal incomes (CEEI). This also holds for the constant-approximate CE.*

The above theorem rules out obtaining polynomial time checkable necessary and sufficient conditions for existence of a CE unless P=NP.[7] The next best hope is to design weakest possible conditions that ensures a CE and captures an interesting class of instances. Towards this we derive two conditions.

The first condition is an artifact of the exchange setting, and is required for dividing goods as well [Max97]: if a set of agents are interested to consume only a strict subset of the endowment that they cumulatively own, then no prices can ensure demand equals supply (we elaborate this shortly in Example 3). We now define a condition that helps us resolve this issue.[8] To define the condition, we first define the *economy graph* of a given instance of chore division.

Definition 2 (Economy Graph [Max97]). *Given an instance $I = \langle A, B, d(\cdot, \cdot), w(\cdot, \cdot) \rangle$, an Economy Graph $G = (A, E)$ is a graph, with vertices corresponding to the agents and there exists an edge from a_i to a_j if and only if there exist a chore $c \in B$, such that $w(a_i, c) > 0$ and $d(a_j, c) \neq \infty$.*

Now we define the first condition.

Definition 3 (Condition 1 [Max97]). *The economy graph of the instance is strongly connected.*

Observe that our instance in Example 1 does satisfy Condition 1, yet does not admit a CE. The primary reason for non-existence of CE in Example 1 is that sets $\{b \in B \mid d(a_1, b) \neq \infty\}$ and $\{b \in B \mid d(a_2, b) \neq \infty\}$ are neither same nor disjoint. Next by generalizing this example we demonstrate that unless finite disutility chore sets of any two agents are either same or disjoint, the equilibrium may not exist. In particular, given any integer $n > 1$ and $m > 1$, we create a chore division instance with n agents and m chores that satisfies Condition 1, has exactly one agent-chore pair with infinite disutility, and does not admit a CE.

[7] In turn there is no unique condition that ensures existence of CE.

[8] In fact, Condition 1 is the analogue of the necessary and sufficient condition required for a CE to exist in exchange markets with goods.

Example 2. *There are n agents $a_1, a_2, \ldots, a_n$, and m chores $b_1, b_2, \ldots, b_m$. We set $w(a_i, b_j) = 1$ for all $i \in [n]$ and $j \in [m]$. So there is a total of n units of each chore b_j, for all $j \in [m]$. Now, we set $d(a_i, b_j) = 1$ for all $i \in [n]$ and $j \in [m-1]$. We set $d(a_i, b_m) = nm$ for all $i \in [n-1]$ and $d(a_n, b_m) = \infty$.*

Since $w(a_i, b_j) = 1$, for all $i \in [n]$ and $j \in [m]$, the instance in Example 2 does satisfy Condition 1 (the economy graph of the instance is a clique). Observe that since all the agents have the same disutility for the chores $\cup_{j \in [m-1]} b_j$, the prices of all these chores will be the same at a CE (otherwise some of the chores will remain unassigned). Therefore, let p be the price of a chore b_j for $j \in [m-1]$, and p' be the price of the chore b_m at a CE. Since a_n has infinite disutility for b_m, she will earn her entire money of $\sum_{j \in [m]} w(a_n, b_j) \cdot p(b_j) = (m-1) \cdot p + p'$ from the chores in $\cup_{j \in [m-1]} b_j$. Therefore, at a CE, the total price of the chores in $\cup_{j \in [m-1]} b_j$ is at least the total money earned by a_n, i.e., total prices of the chores owned by agent a_n, implying that $\sum_{j \in [m-1]} \sum_{i \in [n]} w(a_i, b_j) \cdot p(b_j) \geq \sum_{j \in [m]} w(a_n, b_j) \cdot p(b_j)$. This implies that $(m-1) \cdot n \cdot p \geq (m-1) \cdot p + p'$, further implying that $(m-1) \cdot (n-1) \cdot p \geq p'$. In that case observe that the disutility to price ratio of b_m is strictly less than that of b_1 for any agent a_i, for $i \in [n-1]$: $d(a_i, b_1)/p(b_1) = 1/p \leq ((n-1) \cdot (m-1))/p' < nm/p' = d(a_i, b_m)/p(b_m)$. Thus, none of the agents are willing to do chore b_m, and it remains unassigned, a contradiction.

Our next condition is to circumvent the primary issue in Example 2 that renders CE to not exist. To this end, we define the disutility graph $D = (A \cup B, E_D)$ as the bipartite graph with the set of agents A and the set of chores B forming the vertex sets on two sides and there is an edge from an $a \in A$ to a $b \in B$ when $d(a, b) \neq \infty$. Examples 1 and 2 demonstrate that whenever there is a connected component D' of D which is not a biclique, there exists disutility values for which the instance will not admit a CE. This brings us to our second condition.

Definition 4 (Condition 2)**.** *The disutility graph is a disjoint union of bicliques.*

The second main result of our paper shows that Conditions 1 and 2 guarantee the existence of a CE.

Theorem 2. *A chore division instance satisfying Conditions 1 and 2 admits a CE.*

We now quickly show that even if one of the two conditions is not satisfied, the instance may not admit a CE. Examples 1 and 2 already outline instances that satisfy Condition 1, but do not satisfy Condition 2 and as a result do not admit a CE. We next give an example that satisfies Condition 2, but not Condition 1, and does not admit a CE.

Example 3. *There are three agents a_1, a_2, a_3 and three chores b_1, b_2, b_3. Agents a_1 and a_2 own $1/2$ units of chores b_1 and b_2 each, i.e., $w(a_i, b_j) = 1/2$ for all $i, j \in [2]$. Agent a_3 owns one unit of b_3, i.e., $w(a_3, b_3) = 1$. We set $d(a_1, b_1) = d(a_2, b_1) = 1$, and $d(a_3, b_2) = d(a_3, b_3) = 1$. The disutility value of all other agent chore pair is infinity.*

Observe that the disutility graph is a disjoint union of bicliques – one biclique comprising of agents a_1, a_2 and the chore b_1, and the second biclique comprising of the agent a_3 and chores b_2 and b_3. Therefore the instance satisfies Condition 2. We now show that the instance does not admit a CE. Let $p(b_1)$, $p(b_2)$ and $p(b_3)$ denote the prices of chores b_1, b_2 and b_3 at a CE. Since agents a_1 and a_2 earn their entire endowment money from the chore a_1, we have that $\sum_{i \in [2]} \sum_{j \in [3]} w(a_i, b_j) \cdot p(b_j) = \sum_{i \in [3]} w(a_i, b_1) \cdot p(b_1)$, implying that $p(b_1) + p(b_2) = p(b_1)$, further implying that $p(b_2) = 0$. Therefore, at any CE b_2 will remain unassigned as it will not be a part of the optimal bundle set of the agent a_2 when $p(b_2) = 0$, which is contradiction.

In the subsections that follow, we briefly elaborate our techniques and novel ideas used to prove Theorem 2.

1.2.1 Existence of a CE under the Sufficient Condition

In this section, we sketch the approach and main ideas to show existence of a CE assuming the instance satisfies two sufficient conditions, that is proof of Theorem 2 (see Section 2 for the details). Most equilibrium existence results [Nas51, AD54] are based on either Brouwer's or Kakutani's fixed-point theorems. The Brouwer's (Kakutani's) fixed-point theorem says that given a function (correspondence) ϕ from D to itself, there exists an $x \in D$ such that $f(x) = x$ ($x \in f(x)$), if f is continuous (has closed graph) and D is convex and compact [Bro11, Kak41]. Our proof invokes both Brouwer's and Kakutani's fixed-point theorems, the former nested inside the latter. This approach may be of independent interest to prove existence in other settings.

We first briefly discuss why existence proofs for determining a CE with goods do not easily extend to chores, and this will eventually lead us to the new approach. Most existence proofs for a CE with goods define a fixed-point formulation on the domain of prices that forms a simplex [AD54, Max97], i.e., if there are m goods, then the domain is the simplex $\Delta_m = \{p \in \mathbb{R}^m_{\geq 0} \mid \sum_{j=1}^m p_j = 1\}$. Given the prices, it computes the optimal bundles of agents and adjusts prices based on *excess demand*. At a fixed-point, no change in prices will imply no excess demand, leading to a CE.

This approach immediately fails for the chore division problem due to the issue of *infeasible optimal bundle*: Given a price vector from the simplex domain, if agent a_i's chore endowment has positive total monetary cost, while the chores she is able to do have zero prices, then there is no way she can earn enough money to pay for her chores, in turn making the set $F_i(p)$ in (1) empty. The reason why this issue does not arise in case of goods is that, there, agents are allowed to spend *at most* the total price of their endowments (for bads it is *at least*), thereby reversing the inequality in the definition of the set $F_i(p)$, which ensures that the all zero vector in $\mathbb{R}^m_{\geq 0}$ is always a feasible vector.

To circumvent the above issue, first we need to work with a more involved price domain that ensures that total monetary cost of the chores and endowments is the same inside every component of the disutility graph. Recall the bipartite disutility graph $D = (A \cup B, E_D)$ where there is an edge $(a, b) \in E_D$ if and only if $d(a, b) \neq \infty$. Let $D_1 = (A_1 \cup B_1, E_{D_1}), D_2 = (A_2 \cup B_2, E_{D_2}), \ldots, D_d = (A_d \cup B_d, E_{D_d})$ be the connected components of D. Then, our new price domain is,

$$\mathbf{P} = \left\{ p \in \mathbb{R}^m_{\geq 0} \mid \sum_{j \in [m]} p(b_j) = 1 \text{ and} \sum_{b \in B_k} p(b) = \sum_{a \in A_k} \sum_{j \in [m]} w(a, b_j) p(b_j) \ \ \forall k \in [d] \right\} \tag{2}$$

Now observe that if for any agent $a \in A_k$, for some $k \in [d]$, the chores she is interested in (the set B_k), have zero prices, then the total price of her endowment is also zero as $p \in \mathbf{P}$. In this case, agent a need not earn anything. As a result, she does not need to do any chore and the all zero vector in $\mathbb{R}^m_{\geq 0}$ is a feasible optimal chore set for agent a. Therefore, for any price vector $p \in \mathbf{P}$, for any agent i, we have that the set $F_i(p)$ is not empty and neither is the optimal bundle set in (1). However, there is still an issue with zero prices, a different one: It can be the case that for some component $(A_k \cup B_k, E_k)$, the prices of all the chores in B_k are zero, and prices of all the chores that agents in A_k bring are also zero. In that case, the optimal bundle of any agent $a \in A_k$ consists of only the all zero vector because none of them have to earn anything! However, this will make the optimal bundle set change *non-continuously* with respect to prices, which is a major roadblock in proving continuity like property (the closed graph property) for the fixed-point formulation: for instance consider a simple scenario where there is a component D_k in the disutility graph comprising of just one agent a and one chore b. Agent a has some positive endowment of only one chore $b' \neq b$, say $w(a, b') = 1$ and $w(a, j) = 0$ for all other $j \in B$. Now, consider a sequence of price-vectors $(p_n)_{n \in \mathbb{N}}$ in $\mathbf{P}$, such that $p_n(b') = p_n(b) = 1/n$. Observe that for every $n \in \mathbb{N}$, the optimal bundle of agent a is $X_{ab} = 1$ and $X_{at} = 0$ for all other $t \in B$, as the only chore a is interested in is b, and she has to do one unit of b, to earn her money of $w(a, b) \cdot p(b') = 1 \cdot (1/n) = 1/n$. However, at the limit of the sequence

$(p_n)_{n\in\mathbb{N}}$, say p_*, we have $p_*(b) = p_*(b') = 0$ and the only unique optimal bundle for agent a is the all zero vector in $\mathbb{R}^m_{\geq 0}$. Thus, the optimal bundle may not change continuously with the price-vectors in $\mathbf{P}$.

To fix the above issue, we define the *extended optimal bundle* set, which is same as the optimal bundle set of an agent $a_i \in A_k$, if the total price of the chores in B_k is strictly positive, otherwise it is the set of all feasible allocations of chores in B_k. This will help us ensure *continuity* of the final correspondence. However, we will have to make sure that at the fixed-point, the *extended optimal bundle* is the *optimal bundle* for every agent (one way to do this is to ensure that there are no zero prices at the fixed point). For the allocations, we will work with the following domain: for some sufficiently large constant C, we define

$$\mathbf{X} = \{X \in \mathbb{R}^{mn}_{\geq 0} \mid 0 \leq X_{ij} \leq C, \ \forall a_i \in A, \forall b_j \in B\} \tag{3}$$

Then the set of extended optimal bundles of an agent $a_i \in A_k$ is:

$$EOB_i(p) = \begin{cases} \{X_i \in \mathbf{X} \mid X_{ij} > 0 \text{ only if } d(a_i, b_j) \neq \infty\} & \text{if } \sum_{b \in B_k} p(b) = 0, \\ OB_i(p) & \text{otherwise.} \end{cases} \tag{4}$$

Fixed-point formulation. The domain of our fixed point formulation is $\mathbf{S} = \mathbf{P} \times \mathbf{X}$. Next, we define a correspondence $\phi : \mathbf{S} \to 2^{\mathbf{S}}$ that is product of two correspondences $\phi_1 : \mathbf{S} \to 2^{\mathbf{P}}$ and $\phi_2 : \mathbf{S} \to 2^{\mathbf{X}}$. For a given $(p, X) \in \mathbf{S}$, $\phi(p, X) = \phi_1(p, X) \times \phi_2(p, X)$. Out of these, $\phi_2(p, X)$ is the set of extended optimal bundles at prices p. Formally,

$$\phi_2(p, X) = \{X \in \mathbf{X} \mid X_i \in EOB_i(p), \ \forall a_i \in A\}$$

The exact formulation of ϕ_1 is involved and requires to invoke Brouwer's fixed-point theorem. Therefore, let us first state the properties of ϕ_1 that we need to ensure, and discuss how they help us map fixed-points of ϕ to the competitive equilibria of the chore division instance. For a given $(p, X) \in \mathbf{S}$, if $p' \in \phi_1(p, X)$, then it must be that

- $p' \in P$ and for all components $D_k = (A_k \cup B_k, E_k)$ of the disutility graph, and chores b_j and $b_{j'}$ in B_k, where $p(b_{j'}) > 0$, we have

$$\frac{p'(b_j)}{p'(b_{j'})} = \frac{p(b_j) + max(\sum_{i\in[n]} w(a_i, b_j) - \sum_{i\in[n]} X_{ij}, 0)}{p(b_{j'}) + max(\sum_{i\in[n]} w(a_i, b_{j'}) - \sum_{i\in[n]} X_{ij'}, 0)}. \tag{5}$$

Fixed-points to CE. Let (p, X) be a fixed-point of ϕ, i.e., $(p, X) \in \phi(p, X)$. We first show that at any fixed-point, the prices of all the chores are strictly positive. To the contrary, suppose $p(b_j) = 0$ for some $b_j \in B$, and let b_j belong to component $D_k = (A_k \cup B_k, E_{D_k})$ of the disutility graph D. Then, some component of D has chores with both zero and positive prices. Either it is D_k itself, or if all the chores in D_k have zero prices, then using the fact that $p \in \mathbf{P}$, we have $\sum_{a_i \in A_k} \sum_{j\in[m]} w(a_i, b_j) \cdot p(b_j) = \sum_{b_j \in B_k} p(b_j) = 0$. This implies that the prices of all the chores owned by agents in D_k are zero, and some of them must belong to other components due to the *strong connectivity* of the economy graph (Condition 1). Recursing this argument, and also using the fact that sum of all the prices is 1, there must be a component with a zero priced chore, but the sum of prices of the chores in the component is positive, say component $D_\ell = (A_\ell \cup B_\ell, E_{D_\ell})$.

Let b^0 and b^+ be the chores in D_ℓ with $p(b^0) = 0$ and $p(b^+) > 0$. For every agent in $a_i \in A_\ell$, their $EOB_i(p) = OB_i(p)$, since total price of the chores in B_ℓ is positive (by (4)). Since every $a_i \in D_\ell$ has finite disutility for both b^0 and b^+ (due to Condition 2), her disutility-per-buck for b^0 is strictly more than that for b^1. Due to (1), if $X_i \in OB_i(p)$ then $X_{ib^0} = 0$ for all $i \in A_\ell$. Furthermore, every agent $a \notin A_\ell$ has infinite disutility for b^0, we have that $X_{ib^0} = 0$ for all $i \in [n]$. Now given that our correspondence ϕ satisfies (5), and $p(b^0) = 0$ and $p(b^+) > 0$, we have,

$$0 = \frac{p(b^0)}{p(b^+)} = \frac{p(b^0) + max(\sum_{i\in[n]} w(a_i, b^0) - \sum_{i\in[n]} X_{ib^0}, 0)}{p(b^+) + max(\sum_{i\in[n]} w(a_i, b^+) - \sum_{i\in[n]} X_{ib^+}, 0)}$$
$$= \frac{0 + \sum_{i\in[n]} w(a_i, b^0)}{p(b^+) + max(\sum_{i\in[n]} w(a_i, b^+) - \sum_{i\in[n]} X_{ib^+}, 0)}$$
$$> 0, \text{ a contradiction.}$$

Therefore, at a fixed point, there is no chore with a zero price. Now, we briefly describe why fixed-point (p, X) correspond to the prices and allocation at a CE. Let $r_j(X)$ denote the amount of the chore b_j left undone under X, i.e.,

$$r_j(X) = max(\sum_{i\in[n]} w(a_i, b_j) - \sum_{i\in[n]} X_{ij}, 0).$$

Since all chores have positive price at p, extended optimal bundle set of every agent is her optimal bundle set (by (4)) and thereby $X \in \phi_X(p, X)$ ensures that $X_i \in OB_i(p)$ for every agent $a_i \in A$. Now we only need to ensure demand meets supply for every chore. If not, then some chore b_j in component D_k, which is not completed, i.e., $r_j(X) > 0$. Since $p \in \mathbf{P}$, we have that the cumulative price of the endowments of the agents in a component of the disutility graph equals the total price of the chores in the same component. Since every agent spends on their optimal bundle, the cumulative price of the endowments of the agents equals the total earning of that agents in A_k from B_k. Therefore, if one chore b_j is underdone, i.e., $r_j(X) > 0$, then there exists some other chore $b_{j'}$, which is overdone, i.e., $r_{j'}(X) = 0$. Again using (5), we have $\frac{p(b_j)}{p(b_{j'})} = \frac{p(b_j)+r_j(X)}{p(b_{j'})+r_{j'}(X)} > \frac{p(b_j)}{p(b_{j'})+r_{j'}(X)} = \frac{p(b_j)}{p(b_{j'})}$, a contradiction.

Mapping to P and Ensuring Condition (5). Our next task is to define the correspondence ϕ_1, so that for any given $(p, X) \in \mathbf{S}$, (5) holds for every $p' \in \phi_1(p, X)$, and $p' \in \mathbf{P}$. This in fact is the trickiest part of our proof and constitutes the main bulk of our efforts.

To get $p' \in \mathbf{P}$, we need to make sure that the $p' \in \Delta_m$, and for every component D_k of the disutility graph D, total prices of the chores in D_k equals total cost of endowments of agents in D_k. To this end, for every chore b_j in component D_k, let $q(b_j) = p(b_j) + r_j(X)$, where $r_j(X)$ is the non-negative excess supply as defined above, and $\beta_j = \frac{q(b_j)}{\sum_{b\in D_k} q(b)}$. Note that for (5), we want that for any $b_j, b_{j'} \in D_k$ with $p(b_{j'}) > 0$, $\frac{p'(b_j)}{p'(b_{j'})} = \frac{q(b_j)}{q(b_{j'})} = \frac{\beta_j}{\beta_{j'}}$. Thus, if $\tilde{p}_k = \sum_{b\in D_k} p'(b)$ then $p'(b_j)$ must be $\beta_j \tilde{p}_k$. This reduces to one unknown per component of D, namely $\tilde{p}_k$ for each $k \in [d]$.

Next, we write a system of linear equations to compute $\tilde{p}_k$'s such that all the constraints of domain $\mathbf{P}$ are satisfied. The simplex constraints for the prices in $\mathbf{P}$ can be encoded by ensuring $\tilde{p} \in \Delta_d$. Next, for each component D_k, the following constraint imposes total endowment costs of agents in D_k equals total prices of chores in D_k.

$$\sum_{a_i\in A_k} \sum_{k'\in[d]} \sum_{b_{j'}\in B_{k'}} w(a_i, b_{j'}) \cdot (\beta_{j'} \tilde{p}_{k'}) = \sum_{b_j\in B_k} (\beta_j \tilde{p}_k)$$

Let $M(\beta) \in \mathbb{R}^{d\times d}$ denote the matrix of this linear system. Then, our goal becomes to find a vector $v \in \Delta_d$, in the null space of $M(\beta)$. It is not obvious why such a vector should exist. Our high-level approach to show the same is as follows: We can equivalently express the linear system of equations $M(\beta) \cdot v = 0$ as $M'(\beta) \cdot v = v$, where $M'(\beta) = M(\beta) + I$, where I is the identity matrix. *We show that if we define a function $f : \mathbb{R}^d \to \mathbb{R}^d$ as $f(v) = M'(\beta) \cdot v$, then f maps the d-dimensional simplex Δ_d to itself (this is non-trivial).* Restricting f to only the simplex, we get a continuous map $f : \Delta_d \to \Delta_d$ and therefore it has a fixed-point by the Brouwer's fixed-point theorem. At every fixed-point v we have $M'(\beta) \cdot v = v$ implying $M(\beta) \cdot v = 0$. Since $v \in \Delta_d$ we get the vector we needed.

The above scheme will work if the β_js are well defined. However, for a component D_k if $\sum_{b \in D_k} q(b)$ turns out to be zero, then β_js are ill-defined and cause issues with proving continuity like properties of ϕ. To handle this, we define a set of permissible βs, namely,

$$\mathcal{B} = \left\{ \beta \in \mathbb{R}^m_{\geq 0} \mid \forall k \in [d], \quad \begin{array}{ll} \sum_{b_j \in D_k} \beta_j = 1 & \text{if } \sum_{b \in D_k} t(b) = 0 \\ \forall b_j \in D_k,\ \beta_j = \frac{q(b_j)}{\sum_{b \in D_k} q(b)} & \text{otherwise} \end{array} \right\}.$$

And for each $\beta \in \mathcal{B}$, the above process will compute a $p' \in \phi_1(p, X)$. By construction, each of these p''s will satisfy, $p' \in \mathbf{P}$ and equation (5), as needed. However, it is not immediate why such a set of p''s will form a convex set, as required to apply the Kakutani's fixed point theorem.

In fact, to apply the Kakutani's fixed-point theorem, we need to show that the above complex process creates a ϕ, that has closed graph (continuity-like property), and $\phi(p, X)$ is convex for each $(p, X) \in \mathbf{S}$. This again requires involved argument and is formally proved in Lemmas 15 and 16 of Section 2. Then, ϕ is sure to have a fixed-point which maps to CE as discussed above. We refer the reader to Section 2 for a detailed formal discussion of the entire proof.

Our proof technique extends to show existence of a CE for chore division with general monotone convex disutility functions where an *agent can do only a subset of chores* and with arbitrary endowments, under a similar sufficient condition. Thus, our overall approach may be of independent interest to handle more general problems involving chores.

1.3 Further Related Work

The fair division literature is too vast to survey here, so we refer to the excellent books [BT96, RW98, Mou03] and a recent survey article [Mou19], and restrict attention to previous work that appears most relevant.

Most of the work in fair division is focused on allocating goods with a few exceptions of chores [Su99, AS14, BT96, RW98]. The papers [BMSY17, BMSY19] consider the case of mixed manna that contains both goods and bads in the Fisher model and assume all (dis)utility values to be finite. For the goods case, competitive equilibrium maximizes the Nash welfare, i.e., geometric mean of agents' utilities. In case of chores (or mixed manna), [BMSY17] shows that critical points of the geometric mean of agents' disutilities on the (Pareto) efficiency frontier are the competitive equilibrium profiles. By building on this characterization, [BCM21] recently obtained an efficient algorithm to find an approximate competitive equilibrium (FPTAS). For the special case of constantly many agents (or chores), polynomial-time algorithms are known for computing a competitive equilibrium in the Fisher model [BS19, GM20]. In a recent work, [CGMM21] give a simplex-like algorithm for computing a competitive equilibrium in the exchange model.

Organization of the Rest of the Paper. We present our two main results in the upcoming sections. Section 2 contains the sufficient conditions under which a CE always exists and the proof of existence. Section 3 contains the NP-completeness of determining a CE with chores in the Fisher model.

2 Sufficient Conditions for the Existence of CE

In this section, we formulate certain conditions and prove that if any instance of chore division satisfies these conditions, then the instance will admit a CE. The reader is encouraged to read Section 1.2.1 to get an overall picture of the results, ideas and techniques used in this section.

Recall our sufficient conditions:

Condition 1: The economy graph G of the instance is strongly connected, and

Condition 2: D is a disjoint union of bicliques $D_1, D_2, \ldots, D_d$ for some $d \geq 1$.

Let $\mathcal{I}$ denote all the instances of chore division that satisfy Condition 1 and Condition 2. We now show that all instances in $\mathcal{I}$ admit a CE. Consider any instance $I = \langle G, D \rangle \in \mathcal{I}$ such that$G$ is the economy graph of the instance and $D = \cup_{i \in [d]} D_i$, where each $D_i = (A_i \cup B_i, E_{D_i})$ is a complete bipartite graph, disjoint from $D_{i'}$ ($i' \neq i$). For ease of notation,

- we represent our set A of n agents as $[n]$ (we write a_i as i) and the set B of m chores as $[m]$ (we write chore b_j as j),
- we also write p_j to denote the price of chore b_j (instead of $p(b_j)$) and $w_{i,j}$ to represent the agent a_i's initial endowment of chore b_j (instead of $w(a_i, b_j)$), and
- lastly, we also assume without loss of generality that the total endowment of each chore is one: $\sum_{i \in [n]} w_{i,j} = 1$.

Now, we briefly introduce some basic definitions and concepts required to prove the existence of a CE.

Normalized Prices and Bounded Allocations. A price vector $p = \langle p_1, p_2, \ldots, p_m \rangle$ is called a *normalized price vector* if

- $p_j \geq 0$ for all $j \in [m]$,
- $\sum_{j \in [m]} p_j = 1$, and
- $\sum_{i \in A_k} \sum_{j \in [m]} w_{i,j} \cdot p_j = \sum_{j \in B_k} p_j$ for each component D_k in the disutility graph, i.e., sum of prices of chores in D_k equals the sum of total money of the agents in D_k.

Let P be the set of all normalized price vectors. We first show that the set P is non-empty.

Observation 3. *We have $P \neq \emptyset$.*

Proof. Here we will make use of a general fact that will be useful for a proof later as well.

Fact 1. *Let $Z \in \mathbb{R}^{n \times n}$ be a square matrix such that $Z_{ij} \geq 0$ for all $j \neq i$ (all the non-diagonal entries of Z are non-negative) and $\sum_{i \in [n]} Z_{ij} = 0$ for all $j \in [n]$ (column sums are zero), then there exists a vector $t \in \mathbb{R}^n_{\geq 0}$ such that $\sum_{i \in [n]} t_i = 1$ and $Z \cdot t = 0$.*

The proof of this fact can be found at the end of this section. Using this fact, we will outline a proof that P is non-empty. For each component D_k of the disutility matrix, we pick a chore $b_k \in B_k$ and we set $p_j = 0$ for all $j \in B_k \setminus \{b_k\}$. Note that to show that P is non-empty, it suffices to show that there exists a vector $p' = \langle p'_1, p'_2, \ldots, p'_d \rangle$ (intuitively each p'_k corresponds to the price of chore $b_k \in B_k$, i.e., p_{b_k}) such that $p'_k \geq 0$ for all $k \in [d]$, $\sum_{k \in [d]} p'_k = 1$ and we have,

$$\sum_{i \in A_k} \sum_{k' \in [d]} w_{i,b_{k'}} \cdot p'_{k'} - p'_k = 0 \qquad \text{for all } k \in [d] \tag{6}$$

Let W be the coefficient matrix of the system of equations in (6), i.e., $W \cdot p' = 0$ represents the system of equations in (6). Observe that $W_{kk'} = \sum_{i \in A_k} w_{i,b_{k'}}$ if $k \neq k'$ and $W_{kk} = \sum_{i \in A_k} w_{i,b_k} - 1$. Therefore the non-diagonal entries of W are non-negative and also note that the column sum is zero:

$$\begin{aligned}
\sum_{k \in [d]} W_{kk'} &= \sum_{k \in [d]} \sum_{i \in A_k} w_{i,b_{k'}} - 1 \\
&= \sum_{i \in [n]} w_{i,b_{k'}} - 1 \\
&= 1 - 1 && \text{(total endowment of chore } k' \text{ is one)} \\
&= 0.
\end{aligned}$$

Therefore W satisfies all the conditions in Fact 1. Therefore, by Fact 1 there exists a $p' \in \mathbb{R}^d_{\geq 0}$, such that $\sum_{k \in [d]} p'_d = 1$ and $W \cdot p' = 0$. Therefore, P is non-empty. □

Since P is defined by a set of linear equalities and inequalities, P is closed and convex too. Additionally, since $p \in \mathbb{R}^m_{\geq 0}$ and $\sum_{j \in [m]} p_j = 1$ for all $p \in P$, P is compact.

An allocation $X \in \mathbb{R}^{n \times m}_{\geq 0}$, is called a *bounded allocation* if each X_{ij} (quantifies the amount of chore j allocated to agent i) is non-negative and is at most $m \cdot \frac{d_{max}}{d_{min}}$, where d_{max} and d_{min} refer to the largest and smallest finite entry in the disutility matrix. Let $\mathbf{X}$ be the set of all bounded allocations. Observe that the set $\mathbf{X}$ is non-empty, convex and compact. Also, we have that P is non-empty, convex and compact. We define a compact, convex and non-empty subset of $\mathbb{R}^{(m+nm)}$, $S = \bigcup_{p \in P} \bigcup_{X \in \mathbf{X}} \langle p, X \rangle$ [9].

Correspondence ϕ. Our goal is to define a *correspondence* or equivalently a *set valued function* $\phi : S \to 2^S$, such that ϕ has at least one fixed point and any fixed point of ϕ will correspond to a CE. We will first show some properties that if satisfied by ϕ, then ϕ will have at least one fixed point and any fixed point of ϕ will correspond to a CE. Then, we will define a ϕ that satisfies these properties.

Properties. We first make some basic definitions that will help us to state the properties. We call a bounded allocation $Y \in \mathbf{X}$ an *extended optimal allocation at the price vector* p if and only if,

- for all $i \in A_k$, we have $Y_{ij} > 0$ only if $d(i,j) \neq \infty$, and
- for all $i \in A_k$, where $\sum_{j \in B_k} p_j > 0$, we have $Y_{ij} > 0$ only if $\frac{d(i,j)}{p_j} \leq \frac{d(i,\ell)}{p_\ell}$ for all $\ell \in [m]$, and
- for all $i \in A_k$, where $\sum_{j \in B_k} p_j > 0$, we have $\sum_{j \in [m]} Y_{ij} \cdot p_j = \sum_{j \in [m]} w_{i,j} \cdot p_j$.

Let $\mathbf{X}^p \subseteq \mathbf{X}$ denote the set of all extended optimal allocations at the price vector p. Note that in an extended optimal allocation, the only agents that do not get their optimal bundles (defined in Definition 1) are the ones that belong to a component where the sum of prices of all the chores in the component are zero, as in an extended optimal allocation, an agent that belongs to a component where the sum of prices of all the chores is zero, can be allocated any bundle that does not involve her earning from a chore with infinite disutility (and not necessarily her optimal bundle). However, if $p_j > 0$ for all $j \in [m]$, then every extended optimal allocation is also an optimal allocation (where every agent receives their optimal bundles). Right now, it may not be immediate that $\mathbf{X}^p$ is non-empty. However, we show that this is indeed the case, as agents are allowed to consume goods to a significant extent (Y_{ij} is allowed to be as large as $m \cdot \frac{d_{max}}{d_{min}}$).

Lemma 4. *For all $p \in P$, we have $\mathbf{X}^p \subseteq \mathbf{X}$ and $\mathbf{X}^p \neq \emptyset$.*

[9] We abuse notation slightly here: $\langle p, X \rangle$ refers to the $(m + nm)$-dimensional vector $\langle p_1, p_2, \ldots, p_m, X_{11}, X_{12}, \ldots, X_{nm} \rangle$.

Proof. By definition $\mathbf{X}^p \subseteq \mathbf{X}$. Therefore, it suffices to show that it is non-empty. Consider any $p \in P$. Consider any agent a in the component D_k. Let $\mathbf{w}(a) = \sum_{j\in[m]} w_{a,j} \cdot p_j$. If $\mathbf{w}(a) = 0$, then we set $Y_{aj} = 0$ for all $j \in [m]$ and we trivially have $\sum_{j\in[m]} Y_{aj} \cdot p_j = \sum_{j\in[m]} w_{a,j} \cdot p_j = 0$ and $\langle Y_{a1}, \ldots, Y_{am}\rangle$ is an extended optimal bundle for agent a at p (irrespective of whether $\sum_{j\in B_k} p_j > 0$ or not). So assume that $\mathbf{w}(a) > 0$. Since $p \in P$, we have that the sum of prices of the chores in D_k, $\sum_{j\in B_k} p_j = \sum_{i\in A_k} \sum_{j\in[m]} w_{i,j} \cdot p_j \geq \sum_{j\in[m]} w_{aj} \cdot p_j = \mathbf{w}(a) > 0$. This implies that there is at least one chore b in the component D_k such that $p_b \geq \frac{\mathbf{w}(a)}{m}$. Let b' be a chore such that $d(a, b') \neq \infty$, and $\frac{d(a,b')}{p_{b'}} \leq \frac{d(a,\ell)}{p_\ell}$ for all $\ell \in [m]$. This implies that $\frac{d(a,b')}{p_{b'}} \leq \frac{d(a,b)}{p_b}$. Therefore, we have that

$$
\begin{aligned}
p_{b'} &\geq \frac{d(a, b')}{d(a, b)} \cdot p_b \\
&\geq \frac{d_{min}}{d_{max}} \cdot p_b \\
&\geq \frac{d_{min}}{md_{max}} \cdot \mathbf{w}(a).
\end{aligned}
$$

We set $Y_{ab'} = \frac{\mathbf{w}(a)}{p_{b'}}$. Observe that $Y_{ab'} \leq m \cdot \frac{d_{max}}{d_{min}}$. Therefore, Y is a bounded allocation, i.e., $Y \in \mathbf{X}$. Also, note that agent a earns her entire money of $\mathbf{w}(a)$ by doing $Y_{ab'} = \frac{\mathbf{w}(a)}{p_{b'}}$ amount of chore b' such that $d(a, b') \neq \infty$, $\frac{d(a,b')}{p_{b'}} \leq \frac{d(a,\ell)}{p_\ell}$ for all $\ell \in [m]$. Thus, Y is an extended optimal bundle also. Therefore, $\mathbf{X}^p \neq \emptyset$. □

We are now ready to define the properties of ϕ. For any point $\langle p, X\rangle \in S$, consider any point $\langle p', X'\rangle \in \phi(\langle p, X\rangle)$. Then,

- Property $\mathbf{P}_1$: $X' \in \mathbf{X}^p$ and $p' \in P$.
- Property $\mathbf{P}_2$: For any two agents i and j that belong to the same component D_k of the disutility graph D (say $i, j \in A_k$), such that $p_j > 0$, we have

$$\frac{p'_i}{p'_j} = \frac{p_i + max(1 - \sum_{\ell\in[n]} X_{\ell i}, 0)}{p_j + max(1 - \sum_{\ell\in[n]} X_{\ell j}, 0)}.$$

  .
- Property $\mathbf{P}_3$: $\phi(\langle p, X\rangle)$ is non-empty and convex.
- Property $\mathbf{P}_4$: ϕ has a *closed graph*[10].

We will now show that any correspondence ϕ that satisfies $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ will have at least one fixed point and any fixed point will correspond to CE. We first show that ϕ has a fixed point.

Lemma 5. *Consider any correspondence ϕ that satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$. ϕ has a fixed point.*

Proof. By property $\mathbf{P}_1$ we have that if $\langle p', X'\rangle \in \phi(\langle p, X\rangle)$, then $\langle p', X'\rangle \in S$ (as $p' \in P$ and $X' \in \mathbf{X}^p \subseteq \mathbf{X}$). Therefore, $\phi : S \to 2^S$. The set S is non-empty, compact and convex. Furthermore, by properties $\mathbf{P}_3$ and $\mathbf{P}_4$, we have that $\phi(\langle p, X\rangle)$ is non-empty and convex, and ϕ has a closed graph. Therefore, by Kakutani's fixed point theorem, ϕ has a fixed point. □

[10] A correspondence $\phi : X \to 2^Y$ has a *closed graph* if for all sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, with $(x_n)_{n\in\mathbb{N}}$ converging to x and $(y_n)_{n\in\mathbb{N}}$ converging to y, such that $x_n \in X$ and $y_n \in \phi(x_n)$ for all n, we have $y \in \phi(x)$.

Now we show that any fixed point of a correspondence ϕ that satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ gives a CE.

Lemma 6. *Consider any correspondence ϕ that satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$. Consider any fixed point $\langle p, X\rangle$ of ϕ. Then $\langle p, X\rangle$ is a CE.*

Proof. Consider any fixed point $\langle p, X\rangle \in \phi(\langle p, X\rangle)$. By property $\mathbf{P}_1$, it follows that $X \in \mathbf{X}^p$. As mentioned after that the definition of the extended optimal bundle, if we have $p_j > 0$ for all $j \in [m]$, then each agent gets her optimal bundle in $\mathbf{X}^p$. Therefore, to show that p and X correspond to a CE, it suffices to show that $p_j > 0$ for all $j \in [m]$ and $\sum_{i\in[n]} X_{ij} = 1$ for all chores $j \in [m]$. We first show that $p_j > 0$ for all $j \in [m]$. We prove this by contradiction. Let us assume that there are some chores with zero prices. But first, we make an observation that if there are some chores with zero prices, one of the chores will belong to a component, where the sum of prices of all the chores in that component is non-zero.

Claim 7. *Let p be any price vector in P. If there exists some chore j such that $p_j = 0$, then there exists a chore b in the component D_ℓ of the disutility graph such that $p_b = 0$ and $\sum_{j\in B_\ell} p_j > 0$.*

Proof. We prove this claim by contradiction. Assume otherwise: All chores with zero prices only occur in components where the sum of prices of the chores in the component is zero. Let $D_{\ell_1}, D_{\ell_2}, \ldots, D_{\ell_r}$ be the components of the disutility graph where the sum of prices of all the chores in the component are zero, and there are no chores with zero prices in the components $\bigcup_{k\in[d]\setminus\{\ell_1,\ldots,\ell_r\}} D_k$. Since the economy graph G is strongly connected (by Condition 1), there is an edge from some agent in $\bigcup_{k\in[r]} A_{\ell_k}$ to some agent in $\bigcup_{k\in[d]\setminus\{\ell_1,\ldots,\ell_r\}} A_{\ell_k}$, say from an agent $b' \in A_{\ell_{r'}}$ for some $r' \in [r]$, to an agent $\tilde{b} \in A_{\ell_{\tilde{r}}}$ for $\ell_{\tilde{r}} \in [d] \setminus \{\ell_1, \ldots, \ell_r\}$. Since the agent $\tilde{b}$ has finite disutility only for the chores in $B_{\ell_{\tilde{r}}}$, we can conclude that there exists a chore $\tilde{c} \in B_{\ell_{\tilde{r}}}$, such that $w_{b',\tilde{c}} > 0$. Since $\tilde{c} \in B_{\ell_{\tilde{r}}}$, and there are no chores with zero prices in $D_{\ell_{\tilde{r}}}$ (by assumption), we also have $p_{\tilde{c}} > 0$. Then, we have $\sum_{j\in[m]} w_{b',j} \cdot p_j \geq w_{b',\tilde{c}} \cdot p_{\tilde{c}} > 0$, implying that $\sum_{i\in A_{\ell_{r'}}} \sum_{j\in[m]} w_{i,j} \cdot p_j > 0$. However, since $p \in P$, we have that for component $D_{\ell_{r'}}$ of the disutility graph, the sum of prices of the chores in the component equals the sum of prices of the chores owned by the agents in the same component, implying $\sum_{j\in B_{\ell_{r'}}} p_j = \sum_{i\in A_{\ell_{r'}}} \sum_{j\in[m]} w_{i,j} \cdot p_j > 0$, which is a contradiction. □

Thus, let b be a chore in the component D_k of the disutility graph such that $p_b = 0$ and $\sum_{j\in B_k} p_j > 0$. Then, there is at least one chore $b' \in B_k$ such that $p_{b'} > 0$. Since D_k is biclique (by Condition 2), we have that $d(i, b') \neq \infty$ for all $i \in A_k$. This implies that for all agents $i \in A_k$, we have $\frac{d(i,b')}{p_{b'}} < \frac{d(i,b)}{p_b}$. Since $X \in \mathbf{X}^p$, we have that $X_{ib} = 0$, for all $i \in A_k$ and also for all $i \in [n]$ (as $X \in \mathbf{X}^p$ and $X_{ib} > 0$ only if $d(i, b) \neq \infty$ and for all agents in $[n] \setminus A_k$ we have $d(i, b) = \infty$), implying $\sum_{\ell\in[n]} X_{\ell b} = 0$. Since b and b' both belong to the same component D_k, and $p_{b'} > 0$, by Property $\mathbf{P}_2$, we have,

$$\begin{aligned}
\frac{p_b}{p_{b'}} &= \frac{p_b + max(1 - \sum_{\ell\in[n]} X_{\ell b}, 0)}{p_{b'} + max(1 - \sum_{\ell\in[n]} X_{\ell b'}, 0)} \\
&= \frac{0 + 1}{p_{b'} + max(1 - \sum_{\ell\in[n]} X_{\ell b'}, 0)} \\
&\neq 0 \\
&= \frac{p_b}{p_{b'}},
\end{aligned}$$

which is a contradiction. Thus, none of the chores can have zero prices and therefore, we have $p_j > 0$ for all $j \in [m]$.

We now show that $\sum_{i\in[n]} X_{ij} = 1$ for all $j \in [m]$. We prove this also by contradiction. So assume otherwise and for some chore $b \in B_k$ we have $\sum_{i\in[n]} X_{ib} > 1$ (or $\sum_{i\in[n]} X_{ib} < 1$). Note that, since $p \in P$, for the component D_k of the disutility graph, we have,

$$\sum_{j\in B_k} p_j = \sum_{i\in A_k}\sum_{j\in[m]} w_{i,j}\cdot p_j. \tag{7}$$

Also, since $X \in \mathbf{X}^p$ and every component of the disutility graph has non-zero total price of the chores in it, for every agent $i \in A_k$, we have $\sum_{j\in[m]} w_{i,j}\cdot p_j = \sum_{j\in[m]} X_{ij}\cdot p_j = \sum_{j\in B_k} X_{ij}\cdot p_j$. Substituting $\sum_{j\in[m]} w_{i,j}\cdot p_j$ as $\sum_{j\in B_k} X_{ij}\cdot p_j$ in (7) we have,

$$\begin{aligned}\sum_{j\in B_k} p_j &= \sum_{i\in A_k}\sum_{j\in B_k} X_{ij}\cdot p_j\\ &= \sum_{i\in[n]}\sum_{j\in B_k} X_{ij}\cdot p_j\\ &= \sum_{j\in B_k} p_j\cdot(\sum_{i\in[n]} X_{ij})\ .\end{aligned}$$

Therefore, if $\sum_{i\in[n]} X_{ib} > 1$ (or $\sum_{i\in[n]} X_{ib} < 1$) for some $b \in B_k$, then there exists a $b' \in B_k$ such that $\sum_{i\in[n]} X_{ib'} < 1$ (or $\sum_{i\in[n]} X_{ib'} > 1$). This would imply that $\frac{p_b+max(1-\sum_{\ell\in[n]} X_{\ell b},0)}{p_{b'}+max(1-\sum_{\ell\in[n]} X_{\ell b'},0)} < \frac{p_b}{p_{b'}}$ when $\sum_{i\in[n]} X_{ib} > 1$ and $\frac{p_b+max(1-\sum_{\ell\in[n]} X_{\ell b},0)}{p_{b'}+max(1-\sum_{\ell\in[n]} X_{\ell b'},0)} > \frac{p_b}{p_{b'}}$ when $\sum_{i\in[n]} X_{ib} < 1$, which is a contradiction (as $\frac{p_b+max(1-\sum_{\ell\in[n]} X_{\ell b},0)}{p_{b'}+max(1-\sum_{\ell\in[n]} X_{\ell b'},0)} = \frac{p_b}{p_{b'}}$ if $\langle p, X\rangle$ is a fixed point by property $\mathbf{P}_2$). □

Now, it suffices to show that there exists a correspondence ϕ that satisfies all the four properties to show the existence of CE for every instance $I \in \mathcal{I}$. To this end, we first define a correspondence ϕ and show that it satisfies all the four properties.

Finding a Correspondence ϕ that Satisfies all the Properties. Given a $p \in P$ and $X \in \mathbf{X}$, we define the vector $q = \langle q_1, q_2, \ldots, q_m\rangle$ such that

$$q_j(p,X) = p_j + max(1-\sum_{i\in[n]} X_{ij}, 0) \tag{8}$$

We now introduce a variable $\beta_j(p,X)$ for each chore $j \in [m]$. Let $\beta(p,X) = \langle\beta_1(p,X), \beta_2(p,X), \ldots, \beta_m(p,X)\rangle$. We now outline some constraints that $\beta(p,X)$ must satisfy. We have,

$$\beta_j(p,X) \geq 0 \qquad \text{for all } j\in[m], \tag{9}$$

$$\sum_{j\in B_k}\beta_j(p,X) = 1, \qquad \text{for all } k\in[d], \tag{10}$$

$$\beta_j(p,X) = \frac{q_j(p,X)}{\sum_{j'\in B_k} q_{j'}(p,X)} \qquad \text{for all } j,k,\ \text{such that } j\in B_k,\ \text{and } \sum_{j'\in B_k} q_{j'}(p,X) > 0. \tag{11}$$

Let $\mathcal{B}(p,X)$ be set of all $\beta(p,X)$ that satisfy the system of linear equalities and inequalities in 9, 10 and 11. We now show that $\mathcal{B}(p,X)$ is non-empty, convex and compact.

For each $\beta(p,X) \in \mathcal{B}(p,X)$, we introduce a system of linear equations with a variable $\tilde{p}_k$ for each component D_k of the disutility graph D. Let $\tilde{p} = \langle\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_d\rangle$ (recall that d is the number of components of the disutility graph). We now outline a system of linear equations that needs to be satisfied by a vector $\tilde{p}$. As of now, let us think of each $\tilde{p}_k$ as the sum of prices of the chores in the component D_k and $\beta_j(p, X) \cdot \tilde{p}_k$ as the price of each chore $j \in B_k$. With these price meanings in mind, for each component D_k of D, we write the equation (variables being $\bigcup_{k \in [d]} \tilde{p}_k$) that represents the price of the cumulative endowments of the agents of the component equals the total prices of the chores in the same component.

$$\sum_{i \in A_k} \sum_{k' \in [d]} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta_j(p, X) \cdot \tilde{p}_k - \tilde{p}_k = 0. \tag{12}$$

We represent the system of equations in (12) as

$$M(\beta(p, X)) \cdot \tilde{p} = \mathbf{0} \ . \tag{13}$$

We now make some observation about the non-negativity of the non-diagonal entries and the zero column sums of the matrix $M(\beta(p, X))$.

Observation 8. *We have $M(\beta(p, X))_{kk'} \geq 0$ as long as $k \neq k'$ (every non-diagonal entry of $M(\beta, X)$ is non-zero) and $\sum_{k \in [d]} M(\beta(p, X))_{kk'} = 0$ for all $k' \in [d]$ (column sums are zero).*

Proof. We first carefully look at any column $M(\beta(p, X))_{*k'}$ of $M(\beta, X)$. Note that for all $k \neq k'$, we have, $M(\beta, X)_{kk'} = \sum_{i \in A_k} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta_j(p, X)$. We have $M_{kk} = \sum_{i \in A_k} \sum_{j \in B_k} w_{i,j} \cdot \beta_j(p, X) - 1$. Therefore, every non-diagonal entry in $M(\beta(p, X))$ is non-negative. Now we just need to show that $\mathbf{1}^T \cdot M(\beta(p, X))_{*k'} = 0$. Observe,

$$\begin{aligned} \mathbf{1}^T \cdot M(\beta(p, X))_{*k'} &= \sum_{k \in [d]} \sum_{i \in A_k} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta_j(p, X) - 1 \\ &= \sum_{j \in B_{k'}} \beta_j(p, X) \cdot \sum_{k \in [d]} \sum_{i \in A_k} w_{i,j} - 1 \\ &= \sum_{j \in B_{k'}} \beta_j(p, X) \cdot \sum_{i \in [n]} w_{i,j} - 1 \\ &= \sum_{j \in B_{k'}} \beta_j(p, X) - 1 \\ &= 0. \end{aligned}$$

This shows that $\mathbf{1}^T \cdot M(\beta(p, X)) = \mathbf{0}^T$. □

We first make some observations about the solution to the system of equations in (13) (and consequently (12)). Observe that $M(\beta(p, X))$ satisfies all the conditions in Fact 1. Therefore, we have the following Observation.

Observation 9. *For each $\beta(p, X) \in \mathcal{B}(p, X)$, there exists a vector $\tilde{p} \in \mathbb{R}^d_{\geq 0}$, such that $\sum_{j \in [d]} \tilde{p}_j = 1$ and $M(\beta(p, X)) \cdot \tilde{p} = 0$.*

We are now ready to define the correspondence. Given any $\langle p, X \rangle \in S$, we determine the vector $q(p, X)$ as in (8). Let $\mathcal{B}(p, X)$ be the set of all $\beta(p, X)$ that satisfy the set of linear equalities and inequalities in 9, 10 and 11. For each $\beta(p, X) \in \mathcal{B}(p, X)$, let $\tilde{P}(\beta(p, X)) \subseteq R^d_{\geq 0}$ be the set of all vectors that satisfy the conditions in Observation 9. We now define the set $\overline{P}(\beta(p, X)) \subseteq \mathbb{R}^m_{\geq 0}$ as,

$$\overline{P}(\beta(p, X)) = \left\{ \overline{p} \in \mathbb{R}^m_{\geq 0} \mid \overline{p}_j = \beta_j(p, X) \cdot \tilde{p}_k \ \text{ where chore } j \in B_k \text{ and } \tilde{p} \in \tilde{P}(\beta(p, X)) \right\} \tag{14}$$

Given any $\langle p, X \rangle \in S$, we define

$$\phi(\langle p, X \rangle) = \left\{ \langle \overline{p}, X' \rangle \mid \overline{p} \in \overline{P}(\beta(p, X)) \text{ and } \beta(p, X) \in \mathcal{B}(p, X) \text{ and } X' \in \mathbf{X}^p \right\} \ . \tag{15}$$

For the rest of this section, we will now show that ϕ satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$.

ϕ satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$. Now that we have defined the correspondence, we prove that it satisfies all the necessary properties. To this end consider a point $\langle p', X' \rangle \in \phi(\langle p, X \rangle)$.

Lemma 10 (Property $\mathbf{P}_1$). *Let $\langle p', X' \rangle \in \phi(\langle p, X \rangle)$. We have $X' \in \mathbf{X}^p$ and $p' \in P$.*

Proof. We need to show that $p' \in P$ and $X' \in \mathbf{X}^p \subseteq \mathbf{X}$. Note that by the definition of ϕ we have $X' \in \mathbf{X}^p \subseteq \mathbf{X}$. Therefore, we only need to show that $p' \in P$. Given p and X, let $q(p, X)$ be the vector obtained as in (8) and let $\mathcal{B}(p, X)$ be the set of all $\beta(p, X) \in \mathbb{R}^m$ that satisfy the set of linear inequalities and equalities in 9, 10 and 11. By the definition of the correspondence ϕ (Equation 15), we have that $p' \in \overline{P}(\beta'(p, X))$ for some $\beta'(p, X) \in \mathcal{B}(p, X)$. Equation 14 implies that for each chore $j \in B_k$, we have $p'_j = \beta'_j(p, X) \cdot \tilde{p}_k$, where $\tilde{p} \in \tilde{P}(\beta'(p, X))$. Now we make three claims which show that $p' \in P$.

Claim 11. *We have $p'_j \geq 0$ for all $j \in [m]$.*

Proof. Let us consider any chore j that belongs to the component D_k of the disutility graph. We have,

$$p'_j = \beta'_j(p, X) \cdot \tilde{p}_k.$$

$\beta'(p, X)$ satisfies the system of linear inequalities in 9 and thus $\beta'_j(p, X) \geq 0$. Also, $\tilde{p} \in \tilde{P}(\beta'_j(p, X))$ and by the definition of $\tilde{P}(\beta'(p, X))$ we have that $\tilde{p}_k \geq 0$. Thus $p'_j \geq 0$. □

Claim 12. *We have $\sum_{j \in [m]} p'_j = 1$.*

Proof. We have,

$$\begin{aligned} \sum_{j \in [m]} p'_j &= \sum_{k \in [d]} \sum_{j \in B_k} \beta'_j(p, X) \cdot \tilde{p}_k \\ &= \sum_{k \in [d]} \tilde{p}_k \cdot \sum_{j \in B_k} \beta'_j(p, X). \end{aligned}$$

Since $\beta'(p, X)$ satisfies the set of linear equalities in 10, we have that $\sum_{j \in B_k} \beta'_j(p, X) = 1$. Therefore, we have $\sum_{j \in [m]} p'_j = \sum_{k \in [d]} \tilde{p}_k$. Since $\tilde{p} \in \tilde{P}(\beta'(p, X))$, by definition of $\tilde{P}(\beta'(p, X))$, we have that $\sum_{k \in [d]} \tilde{p}_k = 1$ and thus $\sum_{j \in m} p'_j = 1$. □

Claim 13. *For each component D_k of the disutility graph, we have $\sum_{i \in A_k} \sum_{j \in [m]} w_{i,j} \cdot p'_j = \sum_{j \in B_k} p'_j$.*

Proof. We have,

$$\sum_{i \in A_k} \sum_{j \in [m]} w_{i,j} \cdot p'_j = \sum_{i \in A_k} \sum_{k' \in [d]} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta'_j(p, X) \cdot \tilde{p}_{k'} \ .$$

$\tilde{p} \in \tilde{P}(\beta'(p, X))$, and by definition of $\tilde{P}(\beta'(p, X))$, $\tilde{p}$ satisfies (13) and therefore also (12). Thus, we have $\sum_{i \in A_k} \sum_{k' \in [d]} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta'_j(p, X) \cdot \tilde{p}_{k'} = \tilde{p}_k$. Therefore, we have

$$\begin{aligned} \sum_{i \in A_k} \sum_{j \in [m]} w_{i,j} \cdot p'_j &= \tilde{p}_k \\ &= \sum_{j \in B_k} \beta'_j(p, X) \cdot \tilde{p}_k \qquad \text{(as } \sum_{j \in B_k} \beta'_j(p, X) = 1 \text{ by 10)} \\ &= \sum_{j \in B_k} p'_j. \end{aligned}$$

□

This shows that $p' \in P$ and completes the proof. $\square$

Lemma 14 (Property $\mathbf{P}_2$). *Let $\langle p', X'\rangle \in \phi(\langle p, X\rangle)$. For any two agents i and j that belong to the same component D_k of the disutility graph D ($i, j \in A_k$), such that $p_j > 0$, we have $p'_i/p'_j = \big(p_i + max(1 - \sum_{\ell\in[n]} X_{\ell i}, 0)\big)/\big(p_j + max(1 - \sum_{\ell\in[n]} X_{\ell j}, 0)\big)$.*

Proof. Consider any $\langle p', X'\rangle \in \phi(\langle p, X\rangle)$. By the definition of the correspondence ϕ (Equation 15), we have that $p' \in \overline{P}(\beta'(p, X))$ for some $\beta'(p, X) \in \mathcal{B}(p, X)$. Equation 14 implies that for each chore $j \in B_k$, we have $p'_j = \beta'_j(p, X) \cdot \tilde{p}_k$, where $\tilde{p} \in \tilde{P}(\beta'(p, X))$.

Let i, j be two chores in the component D_k of the disutility graph such that $p_j > 0$. Since $p_j > 0$, we have that $q_j(p, X) \geq p_j > 0$. Therefore, $\sum_{j'\in B_k} q_{j'}(p, X) > 0$. This implies that for all $j \in B_k$, we have $\beta'_j(p, X) = q_j(p, X)/(\sum_{j'\in B_k} q_{j'}(p, X))$. Therefore we have,

$$\begin{aligned}
\frac{p'_i}{p'_j} &= \frac{\beta'_i(p, X) \cdot \tilde{p}_k}{\beta'_j(p, X) \cdot \tilde{p}_k} \\
&= \frac{\beta'_i(p, X)}{\beta'_j(p, X)} \\
&= \frac{q_i(p, X)}{q_j(p, X)} \\
&= \frac{p_i + max(1 - \sum_{\ell\in[n]} X_{\ell i}, 0)}{p_j + max(1 - \sum_{\ell\in[n]} X_{\ell j}, 0)} \quad . \qquad \text{(by definition of } q(p, X) \text{ in (8))}
\end{aligned}$$

$\square$

Lemma 15 (Property $\mathbf{P}_3$). *$\phi(\langle p, X\rangle)$ is non-empty and convex.*

Proof. We first show that $\mathbf{X}^p$ is convex. Consider $Y \in \mathbf{X}^p$ and $Y' \in \mathbf{X}^p$. Let $Y'' = \lambda \cdot Y + (1 - \lambda) \cdot Y'$ for some $\lambda \in [0, 1]$. First observe that $0 \leq min(Y_{ij}, Y'_{ij}) \leq Y''_{ij} \leq max(Y_{ij}, Y'_{ij}) \leq m \cdot \frac{d_{max}}{d_{min}}$. Therefore, $Y'' \in \mathbf{X}$. Now to show that $Y'' \in \mathbf{X}^p$, we need to show that,

1. for all $i \in A_k$, we have $Y''_{ij} > 0$ only if $d(i, j) \neq \infty$, and
2. for all $i \in A_k$, where $\sum_{j\in B_k} p_j > 0$, we have $Y''_{ij} > 0$ only if $\frac{d(i,j)}{p_j} \leq \frac{d(i,\ell)}{p_\ell}$ for all $\ell \in [m]$, and
3. for all $i \in A_k$, where $\sum_{j\in B_k} p_j > 0$, we have $\sum_{j\in[m]} Y''_{ij} \cdot p_j = \sum_{j\in[m]} w_{i,j} \cdot p_j$ for all $i \in [n]$.

To this end, note that for all $i \in A_k$, both Y_{ij} and Y'_{ij} are positive, only if $d(i, j) \neq \infty$. Therefore, $Y''_{ij} > 0$ only if $d(i, j) \neq \infty$. Similarly, for all $i \in A_k$, where $\sum_{j\in B_k} p_j > 0$, both Y_{ij} and Y'_{ij} are positive, only if $\frac{d(i,j)}{p_j} \leq \frac{d(i,\ell)}{p_\ell}$ for all $\ell \in [m]$. Therefore $Y''_{ij} > 0$ only if $\frac{d(i,j)}{p_j} \leq \frac{d(i,\ell)}{p_\ell}$ for all $\ell \in [m]$.

Lastly, for all $i \in A_k$, where $\sum_{j\in B_k} p_j > 0$, we have,

$$\begin{aligned}
\sum_{j\in[m]} Y''_{ij} \cdot p_j &= \sum_{j\in[m]} (\lambda \cdot Y_{ij} + (1 - \lambda) \cdot Y'_{ij}) \cdot p_j \\
&= \lambda \cdot (\sum_{j\in[m]} Y_{ij} \cdot p_j) + (1 - \lambda) \cdot (\sum_{j\in[m]} Y'_{ij} \cdot p_j) \\
&= \lambda \cdot \sum_{j\in[m]} w_{i,j} \cdot p_j + (1 - \lambda) \cdot \sum_{j\in[m]} w_{i,j} \cdot p_j \\
&= \sum_{j\in[m]} w_{i,j} \cdot p_j \quad .
\end{aligned}$$

Thus, $Y'' \in \mathbf{X}^p$. Therefore, $\mathbf{X}^p$ is convex. By Lemma 4, we have that $\mathbf{X}^p$ is non-empty as well. Therefore $\mathbf{X}^p$ is convex and non-empty.

Let $P' = \{p \mid p \in \overline{P}(\beta(p,X)) \text{ for } \beta(p,X) \in \mathcal{B}(p,X)\}$. We now show that P' is convex and non-empty. By Observation 9, we have that for each $\beta(p,X) \in \mathcal{B}(p,X)$, $\tilde{P}(\beta(p,X)) \neq \emptyset$ and by definition of $\overline{P}(\beta(p,X))$ (Equation 14), we have that $\overline{P}(\beta(p,X))$ is also non-empty. Therefore, P' is also non-empty. Now we show that P' is convex as well. To this end, consider two price vectors t and t' in P' or equivalently $t \in \overline{P}(\beta(p,X))$ and $t' \in \overline{P}(\beta'(p,X))$. To show convexity of P', it suffices to show that $\lambda \cdot t + (1-\lambda) \cdot t' \in P'$ for all $\lambda \in [0,1]$ or equivalently $\lambda \cdot t + (1-\lambda) \cdot t' \in \overline{P}(\beta''(p,X))$ for some $\beta''(p,X) \in \mathcal{B}(p,X)$. To this end, we observe that for each chore j in the component D_k of the disutility graph, we have $t_j = \beta_j(p,X) \cdot s_k$, where $s \in \tilde{P}(\beta(p,X))$, and $t'_j = \beta'_j(p,X) \cdot s'_k$, where $s' \in \tilde{P}(\beta'(p,X))$. We now define the vectors $\beta''(p,X)$ and $t'' \in \mathbb{R}^m$ as follows: For each chore j in component D_k of the disutility graph, we define

$$\beta''_j(p,X) = \begin{cases} \frac{\lambda \cdot \beta_j(p,X) \cdot s_k + (1-\lambda) \cdot \beta'_j(p,X) \cdot s'_k}{\lambda s_k + (1-\lambda) \cdot s'_k} & \text{if } s_k \neq 0 \text{ or } s'_k \neq 0 \\ \beta_j(p,X) & \text{otherwise,} \end{cases}$$

and

$$t''_j = \beta''_j(p,X) \cdot s''_k,$$

where $s'' = (\lambda \cdot s + (1-\lambda) \cdot s')$. We first observe that $t'' = \lambda \cdot t + (1-\lambda) \cdot t'$: Consider any $j \in B_k$. If both s_k and s'_k are zero, then $s''_k = \lambda \cdot s_k + (1-\lambda) \cdot s''_k = 0$. Therefore, we have

$$\begin{aligned} t''_j &= \beta''(p,X) \cdot s''_k \\ &= 0 \\ &= \beta(p,X) \cdot s_k + \beta'(p,X) \cdot s'_k && (\text{as } s_k = s'_k = 0) \\ &= \lambda \cdot t_j + (1-\lambda) t'_j \end{aligned}$$

When at least one of s_k or s'_k is non-zero, then $s''_k = \lambda \cdot s_k + (1-\lambda) \cdot s'_k \neq 0$ and we have,

$$\begin{aligned} t''_j &= \beta''(p,X) \cdot s''_k \\ &= \frac{\lambda \cdot \beta_j(p,X) \cdot s_k + (1-\lambda) \cdot \beta'_j(p,X) \cdot s'_k}{\lambda s_k + (1-\lambda) \cdot s'_k} \cdot (\lambda s_k + (1-\lambda) \cdot s'_k) \\ &= \lambda \cdot \beta_j(p,X) \cdot s_k + (1-\lambda) \cdot \beta'_j(p,X) \cdot s'_k \\ &= \lambda \cdot t_j + (1-\lambda) \cdot t'_j. \end{aligned}$$

Now it suffices to show that $\beta''(p,X) \in \mathcal{B}(p,X)$ and $s''_k \in \tilde{P}(\beta''(p,X))$ as this will imply that $\lambda t + (1-\lambda)t' = t'' \in \overline{P}(\beta''(p,X))$ for some $\beta''(p,X) \in \mathcal{B}(p,X)$. We first show that $\beta''(p,X) \in \mathcal{B}(p,X)$. Since s_k, s'_k, $\beta_j(p,X)$, and $\beta'_j(p,X)$ are non-negative and $\lambda \in [0,1]$, we have that $\beta''_j(p,X) \geq 0$ for all $j \in [m]$ and thus $\beta''(p,X)$ satisfies the linear inequalities in 9.

Now we show that $\beta''(p,X)$ satisfies the linear equalities in 10. To this end, consider any component D_k of the disutility graph. If $s_k = s'_k = 0$, then we have $\beta''_j(p,X) = \beta_j(p,X)$ for all $j \in B_k$. Therefore, we have $\sum_{j \in B_k} \beta''_j(p,X) = \sum_{j \in B_k} \beta_j(p,X) = 1$ (as $\beta(p,X) \in \mathcal{B}(p,X)$) and we are done. If one of s_k

or s'_k is non-zero, we have,

$$\begin{aligned}\sum_{j\in B_k}\beta''_j(p,X) &= \sum_{j\in B_k}\frac{\lambda\cdot\beta_j(p,X)\cdot s_k+(1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k}{\lambda\cdot s_k+(1-\lambda)\cdot s'_k}\\ &= \frac{\lambda\cdot s_k\cdot\sum_{j\in B_k}\beta_j(p,X)+(1-\lambda)\cdot s'_k\cdot\sum_{j\in B_k}\beta'_j(p,X)}{\lambda\cdot s_k+(1-\lambda)\cdot s'_k}\\ &= \frac{\lambda\cdot s_k+(1-\lambda)\cdot s'_k}{\lambda\cdot s_k+(1-\lambda)\cdot s'_k}\\ &= 1\end{aligned}$$

Finally, we show that $\beta''(p,X)$ satisfies the linear equalities in 11. To this end, consider any component D_k such that $\sum_{j'\in B_k}q_j(p,X)>0$. In this case, we have $\beta_j(p,X)=\beta'_j(p,X)=q_j(p,X)/(\sum_{j'\in B_k}q_{j'}(p,X))$. Now, if $s_k=s'_k=0$, then we have $\beta''_j(p,X)=\beta_j(p,X)=q_j(p,X)/\left(\sum_{j'\in B_k}q_{j'}(p,X)\right)$ and we are done. If one of s_k or s'_k is non-zero we have,

$$\begin{aligned}\beta''_j(p,X) &= \frac{\lambda\cdot\beta_j(p,X)\cdot s_k+(1-\lambda)\cdot\beta'_j(p,X)\cdot s'_k}{\lambda\cdot s_k+(1-\lambda)\cdot s'_k}\\ &= \frac{\lambda\cdot\frac{q_j(p,X)}{\sum_{j'\in B_k}q_{j'}(p,X)}\cdot s_k+(1-\lambda)\cdot\frac{q_j(p,X)}{\sum_{j'\in B_k}q_{j'}(p,X)}\cdot s'_k}{\lambda\cdot s_k+(1-\lambda)\cdot s'_k}\\ &= \frac{q_j(p,X)}{\sum_{j'\in B_k}q_{j'}(p,X)}\cdot\frac{\lambda\cdot s_k+(1-\lambda)\cdot s'_k}{\lambda\cdot s_k+(1-\lambda)\cdot s'_k}\\ &= \frac{q_j(p,X)}{\sum_{j'\in B_k}q_{j'}(p,X)}.\end{aligned}$$

Thus $\beta''(p,X)\in\mathcal{B}(p,X)$. Now, it only suffices to show that $s''\in\tilde{P}(\beta''(p,X))$. Recall that to show that $s''\in\tilde{P}(\beta''(p,X))$, we need to show that $s''_k\geq 0$ for all $k\in[d]$, $\sum_{k\in[d]}s''_k=1$, and for all $k\in[d]$, we have,

$$\sum_{i\in A_k}\sum_{k'\in[d]}\sum_{j\in B_{k'}}w_{i,j}\cdot\beta''_j(p,X)\cdot s''_{k'}-s''_k=0$$

To this end, we first note that since $s\in\tilde{P}(\beta(p,X))$, we have that

- $s_k\geq 0$ for all $k\in[d]$,
- $\sum_{k\in[d]}s_k=1$, and
- $\sum_{i\in A_k}\sum_{k'\in[d]}\sum_{j\in B_{k'}}w_{i,j}\cdot\beta_j(p,X)\cdot s_{k'}-s_k=0$ for all $k\in[d]$.

Analogous conditions are also satisfied by s' as it belongs to $\tilde{P}(\beta'(p,X))$. Now, observe that $s''_k=\lambda\cdot s_k+(1-\lambda)\cdot s'_k\geq 0$ as both s_k and s'_k are non-negative. Similarly,

$$\begin{aligned}\sum_{k\in[d]}s''_k &= \lambda\cdot\sum_{k\in[d]}s_k+(1-\lambda)\cdot\sum_{k\in[d]}s'_k\\ &= \lambda\cdot 1+(1-\lambda)\cdot 1=1.\end{aligned}$$

Finally, we show that $\sum_{i\in A_k}\sum_{k'\in[d]}\sum_{j\in B_{k'}}w_{i,j}\cdot\beta''_j(p,X)\cdot s''_{k'}-s''_k=0$. To this end, let $K=\{k\mid k\in[d]\text{ and }s''_k>0\}$. Note that it suffices to show $\sum_{i\in A_k}\sum_{k'\in K}\sum_{j\in B_{k'}}w_{i,j}\cdot\beta''_j(p,X)\cdot s''_{k'}-s''_k=$

0 for all $k \in [d]$. Also, for all $k \notin K$, we have $s_k = s'_k = 0$ as well. Therefore, we also have $\sum_{i \in A_k} \sum_{k' \in K} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta_j(p, X) \cdot s_{k'} - s_k = 0$ and $\sum_{i \in A_k} \sum_{k' \in K} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta'_j(p, X) \cdot s'_{k'} - s'_k = 0$ for all $k \in [d]$. Now note that for all $k \in [d]$, we have,

$$
\begin{aligned}
&\sum_{i \in A_k} \sum_{k' \in K} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta''_j(p, X) \cdot s''_{k'} - s''_k \\
&= \Big(\sum_{i \in A_k} \sum_{k' \in K} \sum_{j \in B_{k'}} w_{i,j} \cdot \frac{\lambda \cdot \beta_j(p, X) \cdot s_k + (1-\lambda) \cdot \beta'_j(p, X) \cdot s'_k}{\lambda s_k + (1-\lambda) \cdot s'_k} \cdot s''_{k'} \Big) - s''_k \\
&= \Big(\sum_{i \in A_k} \sum_{k' \in K} \sum_{j \in B_{k'}} w_{i,j} \cdot \frac{\lambda \cdot \beta_j(p, X) \cdot s_k + (1-\lambda) \cdot \beta'_j(p, X) \cdot s'_k}{s''_k} \cdot s''_k \Big) - s''_k \\
&= \Big(\sum_{i \in A_k} \sum_{k' \in K} \sum_{j \in B_{k'}} w_{i,j} \cdot (\lambda \cdot \beta_j(p, X) \cdot s_k + (1-\lambda) \cdot \beta'_j(p, X) \cdot s'_k) \Big) - s''_k \\
&= \Big(\sum_{i \in A_k} \sum_{k' \in K} \sum_{j \in B_{k'}} w_{i,j} \cdot (\lambda \cdot \beta_j(p, X) \cdot s_k + (1-\lambda) \cdot \beta'_j(p, X) \cdot s'_k) \Big) - (\lambda \cdot s_k + (1-\lambda) \cdot s'_k) \\
&= \lambda \cdot \Big(\sum_{i \in A_k} \sum_{k' \in K} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta_j(p, X) \cdot s_k - s_k \Big) + (1-\lambda) \cdot \Big(\sum_{i \in A_k} \sum_{k' \in K} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta'_j(p, X) \cdot s'_k - s'_k \Big) \\
&= 0 + 0 = 0.
\end{aligned}
$$

Therefore $s'' \in \tilde{P}(\beta''(p, X))$.

Therefore, we have that both the sets P' and $\mathbf{X}^p$ are non-empty and convex. thus, $\phi(\langle p, X \rangle)$ is also non-empty and convex as it is the Cartesian product of P' and $\mathbf{X}^p$. □

Lemma 16 (Property $\mathbf{P}_4$). *ϕ has a closed graph.*

Proof. Consider a sequence $(\langle p^n, X^n \rangle)_{n \in \mathbb{N}}$ that converges to $\langle p^*, X^* \rangle$ and $\langle p^n, X^n \rangle \in S$ for all n. Similarly, consider the sequence $(\langle r^n, Y^n \rangle)_{n \in \mathbb{N}}$ that converges to $\langle r^*, Y^* \rangle$, such that $\langle r^n, Y^n \rangle \in \phi(\langle p^n, X^n \rangle)$ for all n. To show that ϕ has a closed graph, we need to show that $\langle r^*, Y^* \rangle \in \phi(\langle p^*, X^* \rangle)$. To show that, $\langle r^*, Y^* \rangle \in \phi(\langle p^*, X^* \rangle)$, we need to show,

1. $r^* \in \overline{P}(\beta(p^*, X^*))$, for some $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$, and
2. $Y^* \in \mathbf{X}^{p^*}$.

Proving $r^* \in \overline{P}(\beta(p^*, X^*))$, for some $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$: We first outline the necessary and sufficient condition for any vector p' to be in $\overline{P}(\beta(p, X))$, as this will be useful for our proof.

Observation 17. *$p' \in \overline{P}(\beta(p, X))$ if and only if*

1. *$p' \in P$, and*
2. *for each chore j in component D_k, we have $p'_j = \beta_j(p, X) \cdot \sum_{j \in B_k} p'_j$.*

Proof. We first show the "if" direction. To show that $p' \in \overline{P}(\beta(p, X))$, it suffices to show that for each chore $j \in B_k$, we have $p'_j = \beta_j(p, X) \cdot \tilde{p}_k$, such that $\tilde{p} \in \tilde{P}(\beta(p, X))$. For each component D_k of the disutility graph, let $\tilde{p}_k = \sum_{j \in B_k} p'_j$. Observe that for each chore $j \in B_k$ we have $p'_j = \beta_j(p, X) \cdot \tilde{p}_k$. It now suffices to show that $\tilde{p} = \langle \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_d \rangle \in \tilde{P}(\beta(p, X))$. To this end, observe that $\tilde{p}_k = \sum_{j \in B_k} p'_j \geq 0$ as $p'_j \geq 0$ for all $j \in [m]$ (as $p' \in P$). Furthermore, $\sum_{k \in [d]} \tilde{p}_k = \sum_{j \in [m]} p'_j = 1$ (as $p' \in P$). Now, to show

$\tilde{p} \in \tilde{P}(\beta(p, X))$, it suffices to show that $\tilde{p}$ satisfies the system of equations in (13) or equivalently those in (12). To this end, since $p' \in P$, for each component D_k we have,

$$\sum_{i \in A_k} \sum_{j \in [m]} w_{i,j} \cdot p'_j = \sum_{j \in B_k} p'_j \ .$$

Or equivalently,

$$\sum_{i \in A_k} \sum_{k' \in [d]} \sum_{j \in B_{k'}} w_{i,j} \cdot p'_j = \sum_{j \in B_k} p'_j \ .$$

Substituting every p'_j as $\beta_j(p, X) \cdot \tilde{p}_k$ where chore j is in the component D_k we have,

$$\sum_{i \in A_k} \sum_{k' \in [d]} \sum_{j \in B_{k'}} w_{i,j} \cdot \beta_j(p, X) \cdot \tilde{p}_{k'} = \tilde{p}_k \ .$$

Therefore, $\tilde{p}_k$ satisfies (12). Thus $\tilde{p} \in \tilde{P}(\beta(p, X))$.

Now we show the "only if" direction. So assume $p' \in \overline{P}(\beta(p, X))$. Then, by Claims 11, 12 and 13 we have that $p' \in P$. Also by the definition of $\overline{P}(\beta(p, X))$, we also have that there exists a vector $\tilde{p} = \langle \tilde{p}_1, \ldots, \tilde{p}_d \rangle$ such that for all $j \in [m]$ we have $p'_j = \beta_j(p, X) \cdot \tilde{p}_k$ where D_k is the component in the disutility graph containing chore j. So it just suffices to show that $\tilde{p}_k = \sum_{j \in B_k} p'_j$ for all $k \in [d]$. To this end, observe that,

$$\begin{aligned} \sum_{j \in B_k} p'_j &= \sum_{j \in B_k} \beta_j(p, X) \cdot \tilde{p}_k \\ &= \tilde{p}_k \cdot \sum_{j \in B_k} \beta_j(p, X) \\ &= \tilde{p}_k. && \text{(as } \beta(p, X) \text{ satisfies 10)} \end{aligned}$$

□

We are now ready to show that $r^* \in \overline{P}(\beta(p^*, X^*))$ for some $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$. r^* is the limit of the sequence $(r^n)_{n \in \mathbb{N}}$, p^* is the limit of the sequence $(p^n)_{n \in \mathbb{N}}$, and X^* is the limit of the sequence $(X^n)_{n \in \mathbb{N}}$. Since for all n, $\langle r^n, Y^n \rangle \in \phi(\langle p^n, X^n \rangle)$, we can conclude that each $r^n \in P$. Since the set P is compact (and therefore closed), we have that $r^* \in P$ as well. Now, by Observation 17, it suffices to show that for each chore j in component D_k, we have $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$ for some $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$. To this end, we first define a vector $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$ and then we show that indeed $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$.

- For all chores $j \in B_k$ such that $\sum_{j' \in B_k} q_{j'}(p^*, X^*) > 0$, we set $\beta_j(p^*, X^*) = q_j(p^*, X^*)/(\sum_{j' \in B_k} q_{j'}(p^*, X^*))$.
- For all chores $j \in B_k$ such that $\sum_{j' \in B_k} q_{j'}(p^*, X^*) = 0$ and $\sum_{j' \in B_k} r^*_{j'} > 0$, we set $\beta_j(p^*, X^*) = r^*_j/(\sum_{j' \in B_k} r^*_{j'})$.
- For all chores $j \in B_k$ such that $\sum_{j' \in B_k} q_{j'}(p^*, X^*) = 0$ and $\sum_{j' \in B_k} r^*_{j'} = 0$, we set $\beta_j(p^*, X^*) = 1/|B_k|$.

It can be verified that $\beta(p^*, X^*)$ satisfies all the linear inequalities and equalities in 9, 10 and 11. Therefore, we have $\beta(p^*, X^*) \in \mathcal{B}(p^*, X^*)$. Now it just suffices to show that $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$. To this end, observe that for all chores $j \in B_k$ such that $\sum_{j' \in B_k} q_{j'}(p^*, X^*) = 0$, we already have that $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$: For a chore $j \in B_k$, where $\sum_{j' \in B_k} r^*_{j'} > 0$, we have $r^*_j = \big(r^*_j/(\sum_{j' \in B_k} r^*_{j'})\big) \cdot$

$(\sum_{j' \in B_k} r^*_{j'}) = \beta_j(p^*, X^*) \cdot (\sum_{j' \in B_k} r^*_{j'})$. Similarly, for a chore $j \in B_k$ where $\sum_{j' \in B_k} r^*_{j'} = 0$, we have $r^*_j = 0 = (1/|B_k|) \cdot 0 = \beta_j(p^*, X^*) \cdot (\sum_{j' \in B_k} r^*_{j'})$.

Therefore, it only suffices to show $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$ for chores j, such that $j \in B_k$ and $\sum_{j' \in B_k} q_{j'}(p^*, X^*) > 0$. To this end, consider any chore $j \in B_k$, such that $\sum_{j' \in B_k} q_{j'}(p^*, X^*) > 0$. Let $\delta = \sum_{j' \in B_k} q_{j'}(p^*, X^*) > 0$ and let $0 < \varepsilon \ll \delta/(2n \cdot m)$. Let $S^* \subseteq S$ be the set of all points $\langle p', X' \rangle \in S$, that have a distance of at most ε from $\langle p^*, X^* \rangle$. Observe that for any $\langle p', X' \rangle \in S^*$ we have,

$$\begin{aligned}
\sum_{j' \in B_k} q_{j'}(p', X') &= \sum_{j' \in B_k} \big(p'_{j'} + max(1 - \sum_{i \in [n]} X'_{ij'}, 0)\big) \\
&\geq \sum_{j' \in B_k} \big((p^*_{j'} - \varepsilon) + (max(1 - \sum_{i \in [n]} X^*_{ij'}, 0) - n\varepsilon)\big) \\
&= \sum_{j' \in B_k} \big(p^*_{j'} + max(1 - \sum_{i \in [n]} X^*_{ij'}, 0)\big) - \sum_{j' \in B_k} (n+1)\varepsilon \\
&\geq \sum_{j' \in B_k} q_{j'}(p^*, X^*) - 2nm\varepsilon \\
&= \delta - 2nm\varepsilon \\
&> 0.
\end{aligned}$$

Thus, for all $\langle p', X' \rangle \in S^*$, we have $\sum_{j' \in B_k} q_{j'}(p', X') > 0$, implying that for all $\beta(p', X') \in \mathcal{B}(p', X')$ we have $\beta_j(p', X') = q_j(p', X')/(\sum_{j' \in B_k} q_{j'}(p', X'))$. Since, $\sum_{j' \in B_k} q_{j'}(p', X') > 0$ for all $\langle p', X' \rangle \in S^*$, we have that $\beta_j(p', X')$ is well defined and *continuous* for all $\langle p', X' \rangle \in S^*$. We define $f_j(r, p, X) = r_j - \beta_j(p, X) \cdot \sum_{j' \in B_k} r_{j'}$. Since $\beta_j(p, X)$ is well defined and continuous for all $\langle p, X \rangle \in S^*$, we have that $f_j(r, p, X)$ is well defined and continuous for all $\langle p, X \rangle \in S^*$ and $r \in P$.

Now, consider any $0 < \varepsilon \ll \delta/(2nm)$. Since the sequences $(r^n)_{n \in \mathbb{N}}$ and $(\langle p^n, X^n \rangle)_{n \in \mathbb{N}}$ converge to r^* and $\langle p^*, X^* \rangle$ respectively, there exists a $n'(\varepsilon) \in \mathbb{N}$ such that for all $n > n'(\varepsilon)$, we have $||r^* - r^n||_2 < \varepsilon$ and $||\langle p^*, X^* \rangle - \langle p^n, X^n \rangle||_2 < \varepsilon$. In that case, for all $n > n'(\varepsilon)$, we have $\langle p^n, X^n \rangle \in S^*$. Therefore, $f_j(r^{n'(\varepsilon)+n}, p^{n'(\varepsilon)+n}, X^{n'(\varepsilon)+n})$ is well defined for all $n \in \mathbb{N}$. We define a new sequence $(h^n)_{n \in \mathbb{N}}$, such that $h^n = f_j(r^{n'(\varepsilon)+n}, p^{n'(\varepsilon)+n}, X^{n'(\varepsilon)+n})$. Since $f_j(r, p, X)$ is well defined and continuous for all $\langle p, X \rangle \in S^*$ and $r \in P$, and $\langle p^{n'(\varepsilon)+n}, X^{n'(\varepsilon)+n} \rangle \in S^*$ and $r^{n'(\varepsilon)+n} \in P$ for all $n \in \mathbb{N}$, we have that the limit of the sequence $(h^n)_{n \in \mathbb{N}}$ is $h^* = f_j(r^*, p^*, X^*)$. Again, since $r^n \in \overline{P}(\beta(p^n, X^n))$ for all $n \in \mathbb{N}$, we have by Observation 17 that $h^n = f_j(r^{n'(\varepsilon)+n}, p^{n'(\varepsilon)+n}, X^{n'(\varepsilon)+n}) = r_j^{n'(\varepsilon)+n} - \beta_j(p^{n'(\varepsilon)+n}, X^{n'(\varepsilon)+n}) \cdot \sum_{j' \in B_k} r_{j'}^{n'(\varepsilon)+n} = 0$ for all $n \in \mathbb{N}$. Therefore, the limit of the sequence $(h^n)_{n \in \mathbb{N}}$ is $h^* = 0$. This implies that $f_j(r^*, p^*, X^*) = 0$, further implying that $r^*_j - \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'} = 0$. Thus, we have $r^*_j = \beta_j(p^*, X^*) \cdot \sum_{j' \in B_k} r^*_{j'}$ for all chores j, such that $j \in B_k$, where $\sum_{j' \in B_k} q_{j'}(p^*, X^*) > 0$.

Proving $Y^* \in \mathbf{X}^{p^*}$: To show $Y^* \in X^{p^*}$, we need to show that

1. $Y^* \in \mathbf{X}$,
2. for all $i \in A_k$, we have $Y_{ij} > 0$ only if $d(i, j) \neq \infty$,
3. for all $i \in A_k$, where $\sum_{j \in B_k} p^*_j > 0$, we have $Y_{ij} > 0$ only if $\frac{d(i,j)}{p^*_j} \leq \frac{d(i,\ell)}{p^*_\ell}$ for all $\ell \in [m]$, and
4. for all $i \in A_k$, where $\sum_{j \in B_k} p^*_j > 0$, we have $\sum_{j \in [m]} Y_{ij} \cdot p^*_j = \sum_{j \in [m]} w_{i,j} \cdot p^*_j$ for all $i \in [n]$.

Since $\langle r^n, Y^n \rangle \in \phi(\langle p^n, X^n \rangle)$ for all n, we have that $Y^n \in \mathbf{X}$ for all n. Since $\mathbf{X}$ is compact (and therefore closed), we have that $Y^* \in \mathbf{X}$ as well.

We show part 2 by contradiction. Assume that there exists an $i \in A_k$, where $Y^*_{ij} = \delta > 0$, and $d(i,j) = \infty$. Since the sequence $(Y^n)_{n\in\mathbb{N}}$ converges to Y^*, we know that for every $\varepsilon > 0$, there exists an $n'(\varepsilon) \in \mathbb{N}$ be such that for $n > n'(\varepsilon)$ we have $|Y^*_{ij} - Y^n_{ij}| < \varepsilon$. Choosing a $\varepsilon \ll \delta$, we can ensure that $|Y^*_{ij} - Y^n_{ij}| < \varepsilon$, implying that $Y^n_{ij} \geq \delta - \varepsilon > 0$ for all $n > n'(\varepsilon)$. Therefore $Y^n_{ij} > 0$ for all $n > n'(\varepsilon)$ (while $d(i,j) = \infty$) which contradicts the fact that $Y^n \in \mathbf{X}$.

We show part 3 by contradiction. Consider any agent $i \in A_k$, where $\sum_{\ell \in B_k} p^*_\ell > 0$. Since $\sum_{\ell \in B_k} p^*_\ell > 0$, the chore j such that $\frac{d(i,j)}{p^*_j}$ is minimum has price $p^*_j > 0$. So for contradiction, let us assume that $Y^*_{ij'} = \beta > 0$, and $\frac{d(i,j')}{p^*_{j'}} > \frac{d(i,j)}{p^*_j}(1+\delta)$ for some $\delta > 0$. Since the sequence $(Y^n)_{n\in\mathbb{N}}$ converges to Y^* and p^n converges to p^*, we know that for every $\varepsilon > 0$, there exists an $n'(\varepsilon) \in \mathbb{N}$ be such that for $n > n'(\varepsilon)$ we have $|Y^*_{ij} - Y^n_{ij}| < \varepsilon$ and $|p^*_j - p^n_j| < \varepsilon$ for all $j \in [m]$. For a sufficiently small $\varepsilon > 0$, we can ensure that $Y^*_{ij'} \geq \beta - \varepsilon > 0$ and $\frac{d(i,j')}{p^n_{j'}} > \frac{d(i,j)}{p^n_j}$, contradicting the fact that $Y^n \in \mathbf{X}^{p^n}$ for all $n > n'(\varepsilon)$.

Finally, we prove part 4 by contradiction. Assume that $\sum_{j\in[m]} w_{i,j} \cdot p^*_j - \sum_{j\in[m]} Y^*_{ij} \cdot p^*_j = \delta$ for some non-zero δ. Since the sequence $(Y^n)_{n\in\mathbb{N}}$ converges to Y^* and p^n converges to p^*, we know that for every $\varepsilon > 0$, there exists an $n'(\varepsilon) \in \mathbb{N}$ be such that for $n > n'(\varepsilon)$ we have $|Y^*_{ij} - Y^n_{ij}| < \varepsilon$ and $|p^*_j - p^n_j| < \varepsilon$ for all $j \in [m]$. Therefore, by choosing a sufficiently small ε we can ensure that $\sum_{j\in[m]} w_{i,j} \cdot p^n_j - \sum_{j\in[m]} Y^n_{ij} \cdot p^n_j \neq 0$, for all $n > n'$, which contradicts the fact that $Y^n \in \mathbf{X}^{p^n}$ for all $n > n'(\varepsilon)$. □

We are now ready to state the main result of this section

Theorem 18. *Every instance $I \in \mathcal{I}$ admits a CE.*

Proof. We defined a correspondence ϕ that satisfies properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ by Lemmas 10, 14, 15 and 16. By Lemma 5 we have that any correspondence that satisfies the properties $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ has a fixed point. Finally, by Lemma 6, any fixed point of this correspondence will correspond to CE in I. Therefore, our correspondence ϕ has at least one fixed point and this fixed point corresponds to a CE. □

Proof of Fact 1: Recall Fact 1.

Fact. *Let $Z \in \mathbb{R}^{n\times n}$ be a square matrix such that $Z_{ij} \geq 0$ for all $j \neq i$ (all the non-diagonal entries of Z are non-negative) and $\sum_{i\in[n]} Z_{ij} = 0$ (column sums are zero), then there exists a vector $t \in \mathbb{R}^n_{\geq 0}$ such that $\sum_{i\in[n]} t_i = 1$ and $Z \cdot t = 0$.*

Proof. Let $\lambda \gg max_{i,j\in[n]}(|Z_{ij}|)$. Let $Z' = \frac{1}{\lambda} Z$. Observe that every t that satisfies $Z' \cdot t = 0$, also satisfies $Z \cdot t = 0$ and vice versa. Also, each entry in the matrix Z' has absolute value less than one. Let $Z'' = (Z' + I)$ where I is the identity matrix. Note that every entry in the matrix Z'' is non-negative. Also every t that satisfies $Z'' \cdot t = t$, also satisfies $Z' \cdot t = 0$ and therefore also satisfies $Z \cdot t = 0$ and vice versa. From here on, we will be dealing with the following system of equations

$$Z'' \cdot t = t \ . \tag{16}$$

We first observe that the matrix Z'' is column stochastic: For all $j \in [n]$, we have

$$\begin{aligned}
\sum_{i\in[n]} Z''_{ij} &= \sum_{i\in[n]} (\frac{1}{\lambda} \cdot Z_{ij} + I_{ij}) \\
&= \sum_{i\in[n]} \frac{1}{\lambda} \cdot Z_{ij} + 1 \\
&= 0 + 1 && \text{(Column sums are zero in } Z\text{)} \\
&= 1 \ .
\end{aligned}$$

Now, let $\Delta_n = \left\{r \in \mathbb{R}^n_{\geq 0} \mid \sum_{j\in[n]} r_j = 1\right\}$ be the n dimensional simplex. Observe that the set Δ_n is non-empty, convex and compact. We first make a small claim.

Claim 19. *Let $r' = Z'' \cdot r$. If $r \in \Delta_n$ then $r' \in \Delta_n$.*

Proof. Since every entry in the matrix Z'' and every component of the vector r is non-negative, we also have that every component of r' is also non-negative: $r'_j \geq 0$ for all $j \in [d]$. Now observe that

$$\begin{aligned}
\sum_{j\in[n]} r'_j &= \mathbf{1}^T \cdot r' \\
&= \mathbf{1}^T \cdot Z'' \cdot r \\
&= \mathbf{1}^T \cdot r && \text{(as } Z'' \text{ is column stochastic)} \\
&= \sum_{j\in[m]} r_j = 1.
\end{aligned}$$

Thus, $r' \in \Delta_n$. ☐

We define $f : \Delta_n \to \Delta_n$ such that $f(r) = Z'' \cdot r$. Observe that f is also continuous. Thus, by Brouwer's fixed point theorem there is a $t \in \Delta_n$, such that $f(t) = t$ or equivalently $Z'' \cdot t = t$. ☐

3 Complexity of Determining the Existence of CE

In this section, we show that determining whether an *arbitrary* instance of chore division admits a CE is strongly NP-hard. In fact, we show that determining whether an instance admits a good approximation to the CE in the Fisher model is strongly NP-hard. In this light, we formally define the problem of determining an α-approximate CE in the Fisher model.

Definition 5 (α-CE in Chore Division in the Fisher Model)**.** *Given a set of agents $A = \{a_1, a_2, \dots, a_n\}$, chores $B = \{b_1, b_2, \dots, b_m\}$, disutilities $d(\cdot,\cdot)$ and fixed earnings $e(\cdot)$, our goal is to find a price vector $p = \langle p(b_1), p(b_2), \dots, p(b_m)\rangle \in \mathbb{R}^m_{\geq 0}$ and allocation $X = \langle X_1, X_2, \dots, X_n\rangle$, such that*

- *Every agent gets their optimal bundle: $X_i \in OB_i(p)$* [11].
- *All chores are almost completely allocated: $\alpha \cdot \sum_{i\in[n]} w(a_i, b_j) \leq \sum_{i\in[n]} X_{ij} \leq \frac{1}{\alpha} \cdot \sum_{i\in[n]} w(a_i, b_j)$ for all $b_j \in B$.*

We show that finding a $(\frac{11}{12} + \delta)$-CE with chores for any $\delta > 0$ in the Fisher model is strongly NP-hard. This will imply that determining a $(\frac{11}{12}+\delta)$-CE in the exchange setting is also strongly NP-hard. Later, in this section we also extend the method to show NP-hardness for finding a $(\frac{11}{12} + \delta)$-CE even in the CEEI setting. In particular, any polynomial time algorithm that determines whether an instance admits a $(\frac{11}{12} + \delta)$-CE in chore division in the Fisher model implies that 3-SAT is solvable in polynomial time.

We quickly recall the 3-SAT problem:

Problem 1. ***(3-SAT)***
Given*: A set of variables $X = \{x_1, x_2, \dots, x_n\}$ and a set of clauses $\mathbf{C} = \{C_1, C_2, \dots, C_m\}$ where each clause is a disjunction of exactly three* literals[12].

[11] Recall that in the Fisher model, $OB_i(p) = \arg\min_{X_i \in F_i(p)} d_i(X_i)$, where $F_i(p) = \left\{X_i \in \mathbb{R}^m_{\geq 0} \mid \sum_{j\in[m]} X_{ij} \cdot p(b_j) \geq e(a_i)\right\}$

[12] A literal is a variable or the negation of a variable

Find: *An assignment* $A : X \to \{T, F\}$ *such that all the clauses are* satisfied[13] *or output that no such assignment exists.*

Given any instance $I = \langle X, \mathbf{C} \rangle$ of 3-SAT, we will create an instance $E(I)$ of chore division such that for any $\delta > 0$, there exists an $(\frac{11}{12} + \delta)$-CE in $E(I)$ if and only if there exists an assignment A that satisfies all the clauses in $\mathbf{C}$ in I. We first briefly sketch the intuition, before we move to the construction of the gadgets required for our reduction.

Several Disconnected Equilibria. We sketch a very simple scenario that could arise in chore division in the Fisher model: Consider an instance with two agents a_1 and a_2 with a fixed earning of one unit each. The disutility values are given below where a_1 has a disutility of 1 for b_1 and 3 for b_2, while a_2 has a disutility of ∞ for b_1 and 1 for b_2.

	b_1	b_2
a_1	1	3
a_2	∞	1

Let $p = \langle p(b_1), p(b_2) \rangle$ be an equilibrium price vector. Also, throughout this section we use the notation MPB_a to denote the *minimum pain per buck* bundle for agent a at the prices p: a chore $b \in MPB_a$ if and only if $\frac{d(a,b)}{p(b)} \le \frac{d(a,b')}{p(b')}$ for all other chores b' in the instance. Observe that this small instance exhibits exactly two competitive equilibria:

- The first CE is when both $p(b_1)$ and $p(b_2)$ are set to 1. Note that only $MPB_{a_1} = \{b_1\}$ and $MPB_{a_2} = \{b_2\}$. Thus a_1 earns her entire one unit of money from g_1 and a_2 earns her entire one unit of money from g_2.
- The second CE is when a_1 earns from both b_1 and b_2. For this we set $p(b_1)$ to $1/2$ and $p(b_2)$ to $3/2$. Note that $MPB_{a_1} = \{b_1, b_2\}$ and $MPB_{a_2} = \{b_2\}$. Under these prices, a_2 earns her entire money by doing $2/3$ of b_2, and a_1 earns her money by doing all of b_1 and $1/3$ of b_2.

Also, observe that there exists no CE at any other set of prices. This is a striking difference to the scenario with only goods to divide, where all CE exists at a unique price vector. Now, let us introduce another agent a_3 and another chore b_3 in the instance. Let us say that a_3 has a fixed earning of one unit, and both agents a_1 and a_2 have a disutility of ∞ towards b_3. We now discuss two scenarios that may arise depending on a_3's disutility towards the chores

1. a_3 has a disutility of 1 towards b_3 and b_2, and ∞ towards b_1.
2. a_3 has a disutility of 1 towards b_3, $\frac{1}{2}$ towards b_1 and ∞ towards b_2.

We will now show that, at a CE, in scenario 1, $b_2 \notin MPB_{a_1}$ and in scenario 2, $b_2 \in MPB_{a_1}$, suggesting that depending on the valuation of a_3, only one *local* equilibrium among the agents a_1, a_2 and chores b_1 and b_2 is admissible at a CE. Let $p(b_1)$, $p(b_2)$ and $p(b_3)$ denote the prices of chores at an equilibrium. Note that since both a_1 and a_2 have a disutility of ∞ for b_3, they only earn money from b_1 and b_2. Thus $p(b_1) + p(b_2) \ge 2$. Note that in both scenarios b_3 should be in MPB_{a_3} as a_3 is the only agent with finite disutility towards it. Now,

[13] A clause $C_r = \ell_1 \vee \ell_2 \vee \ell_3$, where each ℓ_i is a literal, is satisfied if and only if $A(\ell_i) = T$ for at least one $i \in [3]$.

- In scenario 1: Since $b_3 \in MPB_{a_3}$, we have $\frac{d(a_3,b_3)}{p(b_3)} \leq \frac{d(a_3,b_2)}{p(b_2)}$ or equivalently $\frac{1}{p(b_3)} \leq \frac{1}{p(b_2)}$, implying that $p(b_3) \geq p(b_2)$. This in turn implies that

$$\begin{aligned} p(b_2) + 2 &\leq p(b_2) + (p(b_1) + p(b_2)) && \text{(as } p(b_1) + p(b_2) \geq 2\text{)} \\ &\leq p(b_1) + p(b_2) + p(b_3) && \text{(as } p(b_2) \leq p(b_3)\text{)} \\ &= 3. \end{aligned}$$

  Thus we have $p(b_2) \leq 1$, implying that $p(b_1) \geq 1$. Therefore, we can conclude that $b_2 \notin MPB_{a_1}$ as the disutility to price ratio of b_1 is strictly less than that of b_2 for agent a_1.

- In scenario 2: Since $b_3 \in MPB_{a_3}$, we have $\frac{d(a_3,b_3)}{p(b_3)} \leq \frac{d(a_3,b_1)}{p(b_1)}$, we have $\frac{1}{p(b_3)} \leq \frac{1}{2p(b_1)}$, implying that $p(b_3) \geq 2p(b_1)$. This in turn implies that

$$\begin{aligned} 2p(b_1) + 2 &\leq 2p(b_1) + (p(b_1) + p(b_2)) && \text{(as } p(b_1) + p(b_2) \geq 2\text{)} \\ &\leq p(b_1) + p(b_2) + p(b_3) && \text{(as } 2p(b_1) \leq p(b_3)\text{)} \\ &= 3. \end{aligned}$$

  Thus we have $p(b_1) \leq \frac{1}{2}$, implying that $p(b_2) \geq \frac{3}{2}$. Therefore, the disutility to price ratio of b_2 is at most that of b_1 for agent a_1 and thus we conclude that $b_2 \in MPB_{a_1}$.

Thus, as mentioned earlier, the valuations of the agents outside the local sub-instance, impose a specific local equilibrium (among the two disjoint local equilibria) among the agents a_1, a_2 and chores b_1 and b_2. We will now show that when there are n such local sub-instances (resulting in 2^n disjoint equilibria), finding the correct local equilibria becomes intractable.

3.1 Variable Gadgets

For each variable x_i, we introduce two agents a_1^i and a_2^i and two chores b_1^i and b_2^i. We set

$$\begin{aligned} d(a_1^i, b_1^i) &= 1, & d(a_1^i, b_2^i) &= 3, \\ d(a_2^i, b_1^i) &= \infty, & d(a_2^i, b_2^i) &= 1. \end{aligned}$$

See Figure 1 for an illustration. We set the earnings of both a_1^i and a_2^i to be one, i.e., $e(a_1^i) = e(a_2^i) = 1$. Also, for all $i \in [n]$ agents a_1^i and a_2^i have a disutility of ∞ for all other goods in the instance (that have been introduced and will be introduced by clause gadgets in the next section).

3.2 Clause Gadgets

For each clause $C_r = (\ell_i \vee \ell_j \vee \ell_k)$, where ℓ_i is either the variable x_i or its negation $\neg x_i$, we introduce four agents n_i^r, n_j^r, n_k^r and $\mathbf{n}^r$, and three chores m_i^r, m_j^r, and m_k^r. We define the disutility of the agents as follows: For each literal ℓ_i, if

- $\ell_i = x_i$, then,

$$\begin{aligned} d(n_i^r, b_2^i) &= 1 \quad \text{and} \quad d(n_i^r, m_i^r) = \varepsilon \\ d(\mathbf{n}^r, b_2^i) &= 1 \quad \text{and} \quad d(\mathbf{n}^r, m_i^r) = \varepsilon. \end{aligned}$$

  for some $0 < \varepsilon \ll 1$, but $\frac{1}{\varepsilon} \in \mathcal{O}(1)$.

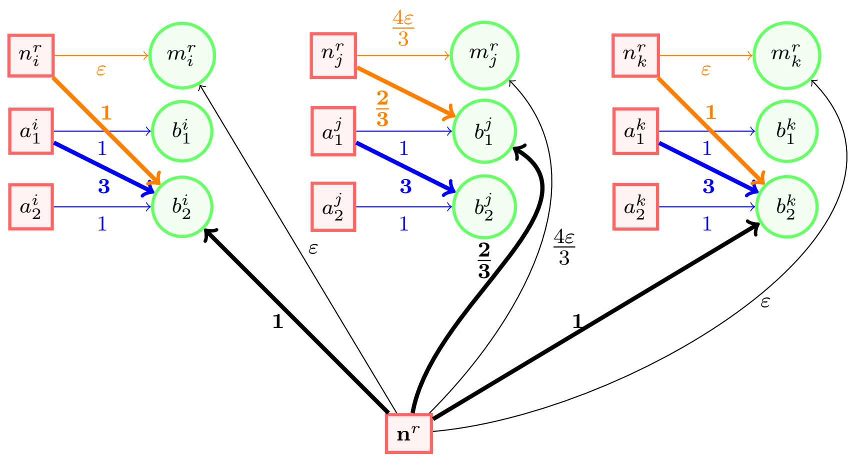

Figure 1: Illustration of the variable gadgets corresponding to x_i, x_j and x_k, and the clause gadget $C_r = (x_i \vee \neg x_j \vee x_k)$. The red squared nodes represent the agents and the green circle nodes represent the chores. Only disutility values less than τ have been indicated. The disutility edges from agents in the variable gadgets are outlined by blue edges. The disutility edges from agents n_ℓ^r for $\ell \in \{i, j, k\}$ are outlined by orange edges and the disutility edges from agent $\mathbf{n}^r$ are outlined by black edges. Thicker disutility edges have a higher disutility than the thinner disutility edges of the same color.

- $\ell_i = \neg x_i$, then,

$$d(n_i^r, b_1^i) = \tfrac{2}{3} \qquad \text{and} \qquad d(n_i^r, m_i^r) = \tfrac{4\varepsilon}{3}$$
$$d(\mathbf{n}^r, b_1^i) = \tfrac{2}{3} \qquad \text{and} \qquad d(\mathbf{n}^r, m_i^r) = \tfrac{4\varepsilon}{3}.$$

For all other agents and chores pair, the disutility is ∞. See Figure 1 for an illustration. We set $e(n_i^r) = e(n_j^r) = e(n_k^r) = \varepsilon$ and $e(\mathbf{n}^r) = \#(C_r) \cdot (\frac{\varepsilon}{2}) + \overline{\#}(C_r) \cdot (\varepsilon) - \varepsilon'$, where $\#(C_r)$ is the number of literals in C_r that are not negations of variables and $\overline{\#}(C_r)$ is the number of literals in C_r that are negations of variables [14], and $\varepsilon' < \frac{\varepsilon}{2}$ (the exact value of ε' will depend on δ [15] and will be made clear in the proof of Lemma 22). We make a small claim about the total earning requirements for the agents n_i^r, n_j^r, n_k^r and $\mathbf{n}^r$.

Claim 20. *For each clause $C_r = \ell_i \vee \ell_j \vee \ell_k$ in I, we have $e(n_i^r) + e(n_j^r) + e(n_k^r) + e(\mathbf{n}^r) = \#(C_r) \cdot (\frac{3\varepsilon}{2}) + \overline{\#}(C_r) \cdot (2\varepsilon) - \varepsilon'$*

Proof. We have,

$$\begin{aligned} e(n_i^r) + e(n_j^r) + e(n_k^r) + e(\mathbf{n}^r) &= 3\varepsilon + \#(C_r) \cdot (\tfrac{\varepsilon}{2}) + \overline{\#}(C_r) \cdot (\varepsilon) - \varepsilon' \\ &= (\#(C_r) + \overline{\#}(C_r))\varepsilon + \#(C_r) \cdot (\tfrac{\varepsilon}{2}) + \overline{\#}(C_r) \cdot (\varepsilon) - \varepsilon' \\ &= \#(C_r) \cdot (\tfrac{3\varepsilon}{2}) + \overline{\#}(C_r) \cdot (2\varepsilon) - \varepsilon' \end{aligned}$$ □

We now show how to map any allocation in $E(I)$ to an assignment of variables in I. Consider any *earning* f under some prices p in $E(I)$. If agent i does X_{ij} amount of chore j, then $f(i, j) = X_{ij} \cdot p(j)$.

If agent a_1^i does some of chore b_2^i, i.e., $f(a_1^i, b_2^i) > 0$, then we set x_i to F and if $f(a_1^i, b_2^i) = 0$, then we set x_i to T.

[14] This implies that $\#(C_r) + \overline{\#}(C_r) = 3$

[15] Reminder to what δ is: recall that we are trying to show the hardness of determining whether an instance admits a $(\frac{11}{12} + \delta)$-CE or not.

We now make some basic observations.

Observation 21. *Let p be the prices of chores and f the the money allocation corresponding to a CE in $E(I)$. Consider any clause $C_r = (\ell_i \vee \ell_j \vee \ell_k)$. Then,*

1. *if $\ell_i = x_i$ and $f(a_1^i, b_2^i) > 0$ then $p(m_i^r) \geq \frac{3\varepsilon}{2}$, and*
2. *if $\ell_i = \neg x_i$ and $f(a_1^i, b_2^i) = 0$, then $p(m_i^r) \geq 2\varepsilon$.*

Proof. We first prove part 1. If $f(a_1^i, b_2^i) > 0$, then $b_2^i \in MPB_{a_1^i}$, implying that $\frac{d(a_1^i,b_2^i)}{p(b_2^i)} \leq \frac{d(a_1^i,b_1^i)}{p(b_1^i)}$. Therefore, we have that $p(b_2^i) \geq \frac{d(a_1^i,b_2^i)}{d(a_1^i,b_1^i)} \cdot p(b_1^i) = 3p(b_1^i)$. Also, note that since agents a_1^i and a_2^i have finite disutility only for chores b_1^i and b_2^i, they will only earn from chores b_1^i and b_2^i. This implies that $p(b_1^i) + p(b_2^i) \geq e(a_1^i) + e(a_2^i) = 2$. Also, since $p(b_2^i) \geq 3p(b_1^i)$ we have that $p(b_2^i) \geq 3/2$. Now, observe that the only agents who have finite disutility towards m_i^r are the agents n_i^r and $\mathbf{n}^r$. Since $\ell_i = x_i$, both n_i^r and $\mathbf{n}^r$ have a disutility of 1 towards b_2^i and ε towards m_i^r. Therefore, for m_i^r to be in either $MPB_{n_i^r}$ or $MPB_{\mathbf{n}^r}$, we need $\frac{\varepsilon}{p(m_i^r)} \leq \frac{1}{p(b_2^i)} \leq \frac{2}{3}$. This implies that $p(m_i^r) \geq \frac{3\varepsilon}{2}$.

The proof of part 2 is very similar. Note that agent a_1^i has finite disutility only for chores b_1^i and b_2^i. If $f(a_1^i, b_2^i) = 0$, then she only earns by doing chore b_1^i, implying that $p(b_1^i) \geq e(a_1^i) = 1$. Similar to the proof in part 1, observe that the only agents who have finite disutility towards m_i^r are the agents n_i^r and $\mathbf{n}^r$. Since $\ell_i = \neg x_i$, both n_i^r and $\mathbf{n}^r$ have a disutility of $\frac{2}{3}$ towards b_1^i and $\frac{4\varepsilon}{3}$ towards m_i^r. Therefore, for m_i^r to be in either $MPB_{n_i^r}$ or $MPB_{\mathbf{n}^r}$, we need $\frac{4\varepsilon}{3p(m_i^r)} \leq \frac{2}{3p(b_1^i)} \leq \frac{2}{3}$ (as $p(b_i^1) \geq 1$). This implies that $p(m_i^r) \geq 2\varepsilon$. ☐

Lemma 22. *If there is no satisfying assignment to the instance $I = \langle X, \mathbf{C} \rangle$ of 3-SAT, then $E(I)$ does not admit any $(\frac{11}{12} + \delta)$-CE for any $\delta > 0$.*

Proof. We prove by contradiction. Assume otherwise and let p be the equilibrium prices of chores and f be the corresponding money allocation. Recall the mapping from an equilibrium allocation to the assignment of variables: For each $i \in [n]$ if $f(a_1^i, b_2^i) > 0$, then we set x_i to F and if $f(a_1^i, b_2^i) = 0$, then we set x_i to T. Since I admits no satisfying assignment, there exists a clause $C_r = \ell_i \vee \ell_j \vee \ell_k$ which is unsatisfied. For every literal $\ell_i \in C_r$ such that $\ell_i = x_i$, note that x_i is F. Therefore, we have that $f(a_1^i, b_2^i) > 0$. This implies that $p(m_i^r) \geq \frac{3\varepsilon}{2}$ (by Observation 21). Similarly for every literal ℓ_i in C_r such that $\ell_i = \neg x_i$, note that x_i is T. Therefore, we have that $f(a_1^i, b_2^i) = 0$, implying that $p(m_i^r) \geq 2\varepsilon$ (by Observation 21). We write the price of chore m_i^r, $p(m_i^r)$ as $\frac{3\varepsilon}{2} + \delta(m_i^r)$ if $\ell_i = x_i$ and $2\varepsilon + \delta(m_i^r)$ if $\ell_i = \neg x_i$, where $\delta(m_i^r)$ is the deviation of the price of m_i^r from its lower bound. Therefore, we have $p(m_i^r) + p(m_j^r) + p(m_k^r) = \#(C_r) \cdot (\frac{3\varepsilon}{2}) + \overline{\#}(C_r) \cdot (2\varepsilon) + \delta(m_i^r) + \delta(m_j^r) + \delta(m_k^r)$. Note that the only agents who have finite disutility for chores m_i^r, m_j^r and m_k^r are the agents n_i^r, n_j^r, n_k^r and $\mathbf{n}^r$. However, by Claim 20, we have that $e(n_i^r) + e(n_j^r) + e(n_k^r) + e(\mathbf{n}^r) = \#(C_r) \cdot (\frac{3\varepsilon}{2}) + \overline{\#}(C_r) \cdot (2\varepsilon) - \varepsilon'$ which is strictly less that the sum of prices of chores m_i^r, m_j^r and m_k^r. In particular we have, $\sum_{h \in \{i,j,k\}} p(m_h^r) - (\sum_{h \in \{i,j,k\}} e(n_h^r) + e(\mathbf{n}^r)) = \varepsilon' + \sum_{h \in \{i,j,k\}} \delta(m_h^r)$. Therefore, there exists at least one chore $m_{h'}^r$ such that the difference between the total price of the chore and the total money earned from the chore by the agents is $\frac{\varepsilon' + \sum_{h \in \{i,j,k\}} \delta(m_h^r)}{3} \geq \frac{\varepsilon' + \delta(m_{h'}^r)}{3}$. Thus, the portion of chore $m_{h'}^r$ left undone is at least,

$$
\begin{aligned}
&= \frac{\varepsilon' + \delta(m_{h'}^r)}{3 \cdot p(m_{h'}^r)} \\
&\geq \frac{\varepsilon' + \delta(m_{h'}^r)}{3 \cdot (2\varepsilon + \delta(m_{h'}^r))} && \text{(as } p(m_{h'}^r) \text{ is either } \tfrac{3\varepsilon}{2} + \delta(m_{h'}^r) \text{ or } 2\varepsilon + \delta(m_{h'}^r)\text{)} \\
&\geq \frac{\varepsilon'}{3 \cdot (2\varepsilon)} && \text{(as } \varepsilon' < \frac{\varepsilon}{2}\text{).}
\end{aligned}
$$

Since our reduction works for any choice of $\varepsilon' < \frac{\varepsilon}{2}$, we can choose an ε' such that $\frac{\varepsilon'}{(6\varepsilon)} > \frac{1}{12} - \delta$, implying that we do not have a $(\frac{11}{12} + \delta)$-CE, which is a contradiction. □

Lemma 23. *If there exists a satisfying assignment to the instance $I = \langle X, \mathbf{C} \rangle$ of 3-SAT, then $E(I)$ admits a CE.*

Proof. Consider any satisfying assignment in I. We now show how to construct the prices p and the money allocation f corresponding to a CE. We will ensure that only the agents in the variable gadgets earn from the chores in the variable gadgets and the agents in the clause gadgets earn only from the chores in the clause gadgets.

Prices and Allocation of Chores in Variable Gadgets. For each variable x_i,

- If $x_i = T$, then we set $p(b_1^i) = 1$ and $p(b_2^i) = 1$.
- If $x_i = F$, then we set $p(b_1^i) = \frac{1}{2}$ and $p(b_2^i) = \frac{3}{2}$.

Since the agents in the variable gadgets have finite disutility only for some goods in the variable gadgets (and have disutility of ∞ for every good in the clause gadget) we can already define their optimal bundles (*MPB* bundles). If $x_i = T$, then observe that $MPB_{a_1^i} = \{b_1^i\}$ and $MPB_{a_2^i} = \{b_2^i\}$. Thus agent a_1^i earns 1 unit of money from doing chore b_1^i entirely and agent a_2^i earns 1 unit of money by doing chore b_2^i entirely. When $x_i = F$, then observe that $MPB_{a_1^i} = \{b_1^i, b_2^i\}$ and $MPB_{a_2^i} = \{b_2^i\}$. Thus agent a_1^i earns 1 unit of money from doing chore b_1^i entirely and b_2^i partly ($1/3$ of chore b_2^i) and agent a_2^i earns 1 unit of money by doing chore b_2^i partly ($2/3$ of chore b_2^i). Now we make an immediate, simple observation:

Observation 24. *When $x_i = T$, then $f(a_1^i, b_2^i) = 0$ and when $x_i = F$, we have $f(a_1^i, b_2^i) > 0$.*

Observe that all the local sub-instances corresponding to the variable gadgets have cleared. It suffices to show that there exists a CE for local sub-instances corresponding to the clause gadgets. We now look into the agents and chores in the clause gadget.

Prices and Allocation of Chores in Clause Gadgets. Consider a clause $C_r = \ell_i \vee \ell_j \vee \ell_k$. Therefore, let $S_r \subseteq \{\ell_i, \ell_j, \ell_k\}$ be the literals that evaluate to T [16] and $U_r \subseteq \{\ell_i, \ell_j, \ell_k\}$ be the set of literals that evaluate to F under the assignment X. Since X is a satisfying assignment, at least one of the literals will evaluate to T and thus $|S_r| \geq 1$ and $|U_r| \leq 2$. Let $\#(S_r)$ and $\#(U_r)$ be the number of literals in S_r and U_r respectively that are not negations of variables and similarly let $\overline{\#}(S_r)$ and $\overline{\#}(U_r)$ be the number of literals that are negations of variables in S_r and U_r respectively. Let α_r be a scalar such that

$$\alpha_r \cdot (\#(U_r) \cdot \tfrac{3\varepsilon}{2} + \overline{\#}(U_r) \cdot (2\varepsilon)) = |U_r| \cdot \varepsilon + e(\mathbf{n}^r) \tag{17}$$

We now set the prices of the chores in the clause gadgets. Consider any clause $C_r = \ell_i \vee \ell_j \vee \ell_k$ in I (with S_r and U_r defined appropriately). For every literal $\ell_\theta \in S_r$, set,

$$p(m_\theta^r) = \begin{cases} \varepsilon & \text{if } \ell_\theta = \neg x_\theta, \\ \varepsilon & \text{if } \ell_\theta = x_\theta \text{ and } U_r \neq \emptyset, \\ \varepsilon + \frac{e(n^r)}{\#(S_r)} & \text{if } \ell_\theta = x_\theta \text{ and } U_r = \emptyset. \end{cases}$$

For every $\ell_\theta \in U_r$, set

[16] A literal $\ell_i = x_i$ evaluates to T if x_i is set to T and the literal $\ell_i = \neg x_i$ evaluates to T when x_i is set to F.

$$p(m^r_\theta) = \begin{cases} \alpha_r \cdot (\frac{3\varepsilon}{2}) & \text{if } \ell_\theta = x_\theta \\ \alpha_r \cdot (2\varepsilon) & \text{if } \ell_\theta = \neg x_\theta. \end{cases}$$

We will now show that under the above prices for the chores in the clause gadgets, we can determine a money flow where all the clause agents earn all of their money from their optimal bundles and all the clause chores will be completed. We distinguish two cases, depending on whether $U_r = \emptyset$ or not,

Case $U_r \neq \emptyset$: In this case, we first observe that α_r is strictly larger than 1:

Observation 25. *We have well defined scalar $\alpha_r > 1$.*

Proof. Since we are in the case where $U_r \neq 0$, we have $\#(U_r) \cdot \frac{3\varepsilon}{2} + \overline{\#}(U_r) \cdot (2\varepsilon) > 0$, thus α_r is well defined. For the claim of the lemma, it suffices to show that $|U_r| \cdot \varepsilon + e(\mathbf{n}^r) > \#(U_r) \cdot \frac{3\varepsilon}{2} + \overline{\#}(U_r) \cdot (2\varepsilon)$. To this end,

$$\begin{aligned} |U_r| \cdot \varepsilon + e(\mathbf{n}^r) &= (\#(U_r) + \overline{\#}(U_r)) \cdot \varepsilon + e(\mathbf{n}^r) \\ &= (\#(U_r) + \overline{\#}(U_r)) \cdot \varepsilon + \#(C_r) \cdot \tfrac{\varepsilon}{2} + \overline{\#}(C_r) \cdot (\varepsilon) - \varepsilon' \ . \end{aligned} \tag{18}$$

Since the literals that are not negations of variables in U_r are also not negations of variables in C_r we have $\#(U_r) \leq \#(C_r)$. By a similar argument we also have $\overline{\#}(U_r) \leq \overline{\#}(C_r)$. Since $|U_r| \leq 2$ we also have $\#(U_r) + \overline{\#}(U_r) < \#(C_r) + \overline{\#}(C_r)$, implying that either $\#(U_r) < \#(C_r)$ or $\overline{\#}(U_r) < \overline{\#}(C_r)$. Therefore, we have that $\#(C_r) \cdot \frac{\varepsilon}{2} + \overline{\#}(C_r) \cdot (\varepsilon) \geq \#(U_r) \cdot \frac{\varepsilon}{2} + \overline{\#}(U_r) \cdot (\varepsilon) + \frac{\varepsilon}{2}$. Plugging this inequality in (18), we have

$$\begin{aligned} |U_r| \cdot \varepsilon + e(\mathbf{n}^r) &\geq (\#(U_r) + \overline{\#}(U_r)) \cdot \varepsilon + \#(U_r) \cdot \tfrac{\varepsilon}{2} + \overline{\#}(U_r) \cdot (\varepsilon) + \tfrac{\varepsilon}{2} - \varepsilon' \\ &> (\#(U_r) + \overline{\#}(U_r)) \cdot \varepsilon + \#(U_r) \cdot \tfrac{\varepsilon}{2} + \overline{\#}(U_r) \cdot (\varepsilon) && (\text{as } \varepsilon' < \tfrac{\varepsilon}{2}) \\ &= \#(U_r) \cdot \tfrac{3\varepsilon}{2} + \overline{\#}(U_r) \cdot (2\varepsilon) \ . \end{aligned}$$
□

We will now characterize the optimal bundles (MPB chores) for each agent under the set prices.

Observation 26. *For each literal $\ell_\theta \in S_r$, we have $m^r_\theta \in MPB_{n^r_\theta}$.*

Proof. We consider the cases, whether the $\ell_\theta = x_\theta$ or $\ell_\theta = \neg x_\theta$.

- $\ell_\theta = x_\theta$: Note that the only other chore (other than m^r_θ) for which agent n^r_θ has finite disutility is chore b^θ_2. Since $\ell_\theta \in S_r$, this means that $x_\theta = T$ and therefore we have $p(b^\theta_2) = 1$ (the way we assigned the prices to the chores in the variable gadgets). Now observe that,

$$\begin{aligned} \frac{d(n^r_\theta, m^r_\theta)}{p(m^r_\theta)} &= \frac{\varepsilon}{\varepsilon} \\ &= 1 \\ &= \frac{d(n^r_\theta, b^\theta_2)}{p(b^\theta_2)} \ . \end{aligned}$$

  Therefore $m^r_\theta \in MPB_{n^r_\theta}$.

- $\ell_\theta = \neg x_\theta$: Note that the only other chore (other than m^r_θ) for which agent n^r_θ has finite disutility is chore b^θ_1. Since $\ell_\theta \in S_r$, this means that $x_\theta = F$ and therefore we have $p(b^\theta_1) = \frac{1}{2}$ (the way we assigned the prices to the chores in the variable gadgets). Now observe that,

$$\begin{aligned}\frac{d(n^r_\theta, m^r_\theta)}{p(m^r_\theta)} &= \frac{4\varepsilon}{3\varepsilon} \\ &= \frac{4}{3} \\ &= \frac{2}{3 \cdot \frac{1}{2}} \\ &= \frac{d(n^r_\theta, b^\theta_1)}{p(b^\theta_1)} \quad .\end{aligned}$$

Therefore, $m^r_\theta \in MPB_{n^r_\theta}$. □

This implies that for all literals ℓ_θ in S_r, the agent n^r_θ will earn her entire money of ε by doing the chore ℓ_θ entirely. Therefore, now we only need to look at the agents n^r_θ and chores m^r_θ where $\ell_\theta \in U_r$. To this end we observe that,

Observation 27. *For each literal $\ell_\theta \in U_r$, we have $m^r_\theta \in MPB_{n^r_\theta}$ and $m^r_\theta \in MPB_{\mathbf{n}^r}$.*

Proof. We first show that $m^r_\theta \in MPB_{n^r_\theta}$. We make a distinction based on whether $\ell_\theta = x_\theta$ or $\ell_\theta = \neg x_\theta$.

- $\ell_\theta = x_\theta$: In this case we have $p(m^r_\theta) = \alpha_r \cdot (\frac{3\varepsilon}{2})$. Note that the only other chore (other than m^r_θ) for which agent n^r_θ has finite disutility is chore b^θ_2. Since $\ell_\theta \in U_r$, this means that $x_\theta = F$ and therefore we have $p(b^\theta_2) = \frac{3}{2}$ (the way we assigned the prices to the chores in the variable gadgets). Now observe that,

$$\begin{aligned}\frac{d(n^r_\theta, m^r_\theta)}{p(m^r_\theta)} &= \frac{1}{\alpha_r} \cdot \frac{\varepsilon}{\frac{3\varepsilon}{2}} \\ &= \frac{1}{\alpha_r} \cdot \frac{2}{3} \\ &= \frac{1}{\alpha_r} \cdot \frac{d(n^r_\theta, b^\theta_2)}{p(b^\theta_2)} \\ &< \frac{d(n^r_\theta, b^\theta_2)}{p(b^\theta_2)} \quad . && \text{(as } \alpha_r > 1 \text{ by Observation 25)}\end{aligned} \tag{19}$$

- $\ell_\theta = \neg x_\theta$: In this case we have $p(m^r_\theta) = \alpha_r \cdot (2\varepsilon)$. Note that the only other chore (other than m^r_θ) for which agent n^r_θ has finite disutility is chore b^θ_1 (the way we assigned the prices to the chores in the variable gadgets). Since $\ell_\theta \in U_r$, this means that $x_\theta = T$ and therefore we have $p(b^\theta_1) = 1$. Now observe that,

$$\begin{aligned}
\frac{d(n^r_\theta, m^r_\theta)}{p(m^r_\theta)} &= \frac{1}{\alpha_r} \cdot \frac{4\varepsilon}{3 \cdot 2\varepsilon} \\
&= \frac{1}{\alpha_r} \cdot \frac{2}{3} \\
&= \frac{1}{\alpha_r} \cdot \frac{d(n^r_\theta, b^\theta_1)}{p(b^\theta_1)} \\
&< \frac{d(n^r_\theta, b^\theta_1)}{p(b^\theta_1)} \ . \qquad \text{(as } \alpha_r > 1 \text{ by Observation 25)}
\end{aligned} \tag{20}$$

Thus, in both cases we have $m^r_\theta \in MPB_{n^r_\theta}$.

We will now show that $m^r_\theta \in MPB_{\mathbf{n}^r}$ as well. We do this by showing that the disutility to price ratio of the chores m^r_θ, when $\ell_\theta \in U_r$, is minimum for the agent $\mathbf{n}^r$. *To this end, first crucially observe that from* (19) *and* (20)*, irrespective of whether* $\ell_\theta = x_\theta$ *or* $\ell_\theta = \neg x_\theta$*, we have* $\frac{d(n^r_\theta, m^r_\theta)}{p(m^r_\theta)} = \frac{1}{\alpha_r} \cdot \frac{2}{3}$. Also, note that the disutility profile agent $\mathbf{n}^r$ has for chore m^r_θ and the chores in the variable gadget of x_θ (b^θ_1 and b^θ_2) is identical to the disutility profile of agent n^r_θ for the same set of chores. Therefore, for all $\ell_\theta \in U_r$ we have $\frac{d(\mathbf{n}^r, m^r_\theta)}{p(m^r_\theta)} = \frac{1}{\alpha_r} \cdot \frac{2}{3}$ (irrespective of whether $\ell_\theta = x_\theta$ or $\ell_\theta = \neg x_\theta$) which is also strictly less than both $\frac{d(\mathbf{n}^r, b^\theta_2)}{p(b^\theta_2)}$ and $\frac{d(\mathbf{n}^r, b^\theta_1)}{p(b^\theta_1)}$. We now look at disutility to price ratio that agent $\mathbf{n}^r$ has for chores in S_r. Observe that for all $\ell_\beta \in S_r$ we have $p(m^r_\beta) = \varepsilon$ and $d(\mathbf{n}^r, m^r_\beta) \geq \varepsilon$ (as the disutility is ε if $\ell_\beta = x_\beta$ and is $\frac{4\varepsilon}{3}$ if $\ell_\beta = \neg x_\beta$). This implies that for all $\ell_\beta \in S_r$ we have $\frac{d(\mathbf{n}^r, m^r_\beta)}{p(m^r_\beta)} \geq 1 > \frac{2}{3} > \frac{1}{\alpha_r} \cdot \frac{2}{3}$ (as $\alpha_r > 1$ by Observation 25). Therefore, the disutility to price ratio of the chores m^r_θ, when $\ell_\theta \in U_r$, for agent $\mathbf{n}^r$ is $\frac{1}{\alpha_r} \cdot \frac{2}{3}$ which is at most the disutility to price ratio of all the chores for which $\mathbf{n}^r$ has finite disutility . Therefore, we have $\bigcup_{\ell_\theta \in U_r} m^r_\theta \subseteq MPB_{\mathbf{n}^r}$. □

Now that we have identified the MPB chores for all the agents in the clause gadgets, we are ready to show the money flow allocation. We set

$$\begin{aligned}
&f(n^r_\theta, m^r_\theta) = \varepsilon && \text{(for all } \ell_\theta \in S_r) \\
&f(\mathbf{n}^r, m^r_\theta) = p(m^r_\theta) - \varepsilon \ . && \text{(for all } \ell_\theta \in U_r)
\end{aligned}$$

All agents spend on their corresponding MPB chores. Observe that for all $\ell_\theta \in S_r$, the agents n^r_θ earn their money of ε by doing chore m^r_θ completely. Now, for all $\ell_\theta \in U_r$, the agents n^r_θ earn their money of ε by doing chore m^r_θ partially. The agent $\mathbf{n}^r$ earns her entire money by completing whatever is left of the chores in $\bigcup_{\ell_\theta \in U_r} m^r_\theta$. It only suffices to show that agent $\mathbf{n}^r$ earns exactly $e(\mathbf{n}^r)$. To this end, we observe that the total money earned by $\mathbf{n}^r$ is

$$\begin{aligned}
\sum_{\ell_\theta \in U_r} f(\mathbf{n}^r, m^r_\theta) &= \sum_{\ell_\theta \in U_r} (p(m^r_\theta) - \varepsilon) \\
&= \alpha_r \cdot (\#(U_r) \cdot \tfrac{3\varepsilon}{2} + \overline{\#}(U_r) \cdot (2\varepsilon)) - |U_r| \cdot \varepsilon \\
&= e(\mathbf{n}^r) \ . \qquad \text{(by (17))}
\end{aligned}$$

Therefore, we have an allocation where the agents in the corresponding variable gadgets earn their money by completing the chores in the variable gadgets and the agents in the clause gadget earn their entire money by completing the chores in the clause gadgets. This concludes the proof for the case $U_r \neq \emptyset$.

Case $U_r = \emptyset$: In this case we have that all the literals in the clause C_r belongs to the set S_r. Therefore, for all the literals ℓ_θ occurring in C_r, we have,

$$p(m^r_\theta) = \begin{cases} \varepsilon & \text{if } \ell_\theta = \neg x_\theta, \\ \varepsilon + \frac{e(n^r)}{\#(S_r)} & \text{if } \ell_\theta = x_\theta \end{cases}$$

Like earlier, we will identify the *MPB* chores for all the clause gadget agents and then will outline a money flow allocation where every agent earns all her money and all the chores are completed. We first look into the agents n^r_θ. Very similar to Observation 26, we can claim that $m^r_\theta \in MPB_{n^r_\theta}$ with a very similar argument as the one used in the proof of Observation 26: The agent n^r_θ has finite disutility only for chores m^r_θ, b^θ_2 if $\ell_\theta = x_\theta$, and only for chores m^r_θ and b^θ_1 if $\ell_\theta = \neg x_\theta$, and the price of the chore $p(m^r_\theta)$ is at least ε (it is more if $\ell_\theta = x_\theta$), while the prices of chores b^1_θ and b^2_θ are the same as in Observation 26.

Now we look into the agent $\mathbf{n}^r$. Since the disutility profile of agent $\mathbf{n}^r$ is identical to that of n^r_θ, when restricted to chores b^θ_1, b^θ_2 and m^r_θ, we can conclude that the disutility to price ratio of m^r_θ for $\mathbf{n}^r$ is at most that of chores b^θ_1 and b^θ_2. Now observe that the disutility to price ratio of all chores m^r_θ for $\mathbf{n}^r$ where $\ell_\theta = x_\theta$ is $\frac{d(\mathbf{n}^r, m^r_\theta)}{p(m^r_\theta)} = \frac{\varepsilon}{p(m^r_\theta)} \leq 1$ (as $p(m^r_\theta) = \varepsilon + \frac{e(\mathbf{n}^r)}{\#(S_r)}$), while the disutility to price ratio all chores m^r_θ for $\mathbf{n}^r$ where $\ell_\theta = \neg x_\theta$ is $\frac{d(\mathbf{n}^r, m^r_\theta)}{p(m^r_\theta)} = \frac{4\varepsilon}{3p(m^r_\theta)} > 1$ (as $p(m^r_\theta) = \varepsilon$). Since $\mathbf{n}^r$ has finite disutility only for the chores in the clause gadget of C_r and the chores in the corresponding variable gadgets, we can claim that $\bigcup_{\{\theta | \ell_\theta = x_\theta\}} m^r_\theta \subseteq MPB_{\mathbf{n}^r}$. Now, that we have identified the *MPB* chores for the agents in the clause gadget, we outline a money flow,

$$\begin{aligned} f(n^r_\theta, m^r_\theta) &= \varepsilon && \text{(for all } \ell_\theta) \\ f(\mathbf{n}^r, m^r_\theta) &= p(m^r_\theta) - \varepsilon \ . && \text{(for all } \ell_\theta = x_\theta) \end{aligned}$$

All the agents spend on their corresponding *MPB* chores. Observe that for all ℓ_θ, the agents n^r_θ earn their entire money of ε by doing chore m^r_θ (partially if $\ell_\theta = x_\theta$ and completely when $\ell_\theta = \neg x_\theta$). The agent $\mathbf{n}^r$ earns her entire money by completing whatever is left of the chores in $\bigcup_{\{\theta | \ell_\theta = x_\theta\}} m^r_\theta$. It only suffices to show that agent $\mathbf{n}^r$ earns exactly $e(\mathbf{n}^r)$. To this end, we observe that the total money earned by $\mathbf{n}^r$ is

$$\begin{aligned} \sum_{\{\theta | \ell_\theta = x_\theta\}} f(\mathbf{n}^r, m^r_\theta) &= \sum_{\{\theta | \ell_\theta = x_\theta\}} (p(m^r_\theta) - \varepsilon) \\ &= \#(S_r) \cdot \left(\varepsilon + \frac{e(\mathbf{n}^r)}{\#(S_r)} - \varepsilon \right) \\ &= e(\mathbf{n}^r). \end{aligned}$$

Therefore, we have an allocation where the agents in the variable gadgets earn their money by completing the chores in the variable gadgets and the agents in the clause gadgets earn their entire money by completing the chores in the clause gadgets. This concludes the proof for the case $U_r = \emptyset$. □

This brings us to the main result of this section.

Theorem 28. *Determining an $(\frac{11}{12} + \delta)$-CE, for any $\delta > 0$, in chore division in the Fisher model is strongly NP-hard.*

Proof. Given any instance $I = \langle X, \mathbf{C} \rangle$ of 3-SAT, in polynomial time we can construct an instance $E(I)$ of chore division comprising of all variable gadgets and clause gadgets. Also, observe all the entries in the disutility matrix $d(\cdot, \cdot)$ and the money vector $e(\cdot)$ are constants (Thus all input parameters can be expressed with polynomial bit size in unary notation). Lemma 22 implies that we have a $(\frac{11}{12}+\delta)$-CE only if I is satisfiable and Lemma 23 implies that if I is satisfiable, then $E(I)$ admits a CE (and thus also a $(\frac{11}{12}+\delta)$-CE). □

Remark 29. *Note that every instance of chore division in the Fisher model $\langle A, B, d(\cdot,\cdot), e(\cdot)\rangle$, where $e(a)$ is an integer for all $a \in A$, can be transformed into an instance $I' = \langle A', B, d'(\cdot,\cdot)\rangle$ of chore division in the CEEI model (where $e(a) = 1$ for all $a \in A'$) by creating $e(a)$ many identical copies (having the exact same disutility profile) of the agent $a \in A$ (the good set remains unchanged): Every α-CE in I' will also be an α-CE in I. Observe that in our instance $E(I)$, we can scale the earning functions of all the agents by some large scalar $\gamma(\varepsilon, \varepsilon')$ to make the earnings of the agents integral. Again, since $e(a) \in \mathcal{O}(1)$ and $\frac{1}{\varepsilon}, \frac{1}{\varepsilon'} \in \mathcal{O}(1)$, we have $|A'| = \mathcal{O}(|A|)$ and all the input parameters of A' (all entries in the disutility matrix $d'(\cdot,\cdot)$) can be expressed with polynomial bit size in unary notation.* Therefore, finding an $(\frac{11}{12}+\delta)$-CE, for any $\delta > 0$, in chore division in the CEEI model is also strongly NP-hard.

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