ETR-Completeness for Decision Versions of Multi-Player (Symmetric) Nash Equilibria *

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Abstract. As a result of some important works [19,6,3,10,5], the complexity of 2-player Nash equilibrium is by now well understood, even when equilibria with special properties are desired and when the game is symmetric. However, for multi-player games, when equilibria with special properties are desired, the only result known is due to Schaefer and Štefankovič [22]: that checking whether a 3-player Nash Equilibrium (3-Nash) instance has an equilibrium in a ball of radius half in l_∞-norm is ETR-complete, where ETR is the class Existential Theory of Reals.

Building on their work, we show that the following decision versions of 3-Nash are also ETR-complete: checking whether (i) there are two or more equilibria, (ii) there exists an equilibrium in which each player gets at least h payoff, where h is a rational number, (iii) a given set of strategies are played with non-zero probability, and (iv) all the played strategies belong to a given set.

Next, we give a reduction from 3-Nash to symmetric 3-Nash, hence resolving an open problem of Papadimitriou [18]. This yields ETR-completeness for symmetric 3-Nash for the last two problems stated above as well as completeness for the class FIXP$_a$, a variant of FIXP for strong approximation. All our results extend to k-Nash, for any constant $k \geq 3$.

1 Introduction

Nash equilibrium (NE) is arguably the most important and well-studied solution concept within game theory and understanding its complexity has led to an impressive theory which was discovered largely over the last decade. We denote by k-Nash the problem of computing a NE in a k-player game for a constant k. For the case of 2-Nash, the seminal results of Daskalakis, Goldberg and Papadimitriou [6], and Chen and Deng [3] exactly characterized the complexity of this problem, namely it is PPAD-complete. This leads us to another basic question: of finding a k-Nash solution that satisfies special properties, e.g., has a payoff of at least h for each player. These questions were first studied by Gilboa and Zemel [10]: they considered 2-Nash under numerous special properties and showed them all to be NP-complete [5]. Thus the complexity of the 2-player case is very well understood.

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Although the 2-player case is the most classical and well studied case, it is also important to study the complexity of the multi-player, especially in the context of new applications arising on the Internet and other large networks where multiple players are locked in strategic situations. Indeed there has been much activity on this front, e.g., see [20,11,1], but the picture is not as clear as the 2-player case. A fundamental difference between 2-Nash and k-Nash, for $k \geq 3$, is that whereas the former always admits an equilibrium that can be written using rational numbers [12], the latter require irrational numbers in general, as shown by Nash himself [15] (we will assume that all numbers in the given instance are rational). It is easy to see that in the latter case, equilibria are algebraic numbers. This difference makes the multi-player case much harder.

Daskalakis, Goldberg and Papadimitriou [6], showed that for k-player games, $k \geq 3$, finding an ϵ-approximate Nash equilibrium is PPAD-complete. The complexity of exact equilibrium was resolved by Etessami and Yannakakis [7], who showed this case to be complete for their class FIXP. How about the complexity of finding a k-Nash solution that satisfies special properties? Due to the inherent difficulty of dealing with irrational numbers, this problem remained open until 2009, when Schaefer and Štefankovič [22] formally defined class *Existential Theory of Reals* (ETR), and showed that checking if a 3-player game has a NE in which every strategy is played with probability at most 0.5 (**InBox**) is ETR-complete. ETR is the class of "yes" instances of existentially quantified formulas with bases $\{+, -, *, \wedge, \vee, =, <, >\}$ on real numbers; we note that this class was informally known and used earlier than [22], e.g., see [2].

Our first set of results extend ETR-completeness to NE computation with a number of special properties in ≥ 3 player games: (i) checking if a game has more than one NE (**NonUnique**). NE where, (ii) each player gets at least h payoff (**MaxPayoff**), (iii) a given set of strategies are played with +ve probability (**Subset**), or (iv) all the played strategies belong to a given set (**Superset**).

Our second set of results deal with symmetric games. Symmetry arises naturally in numerous strategic situations and with the growth of the Internet, on which typically users are indistinguishable, such situations are only becoming more ubiquitous. In a *symmetric game* all players participate under identical circumstances, i.e., strategy sets and payoffs. Thus the payoff of player i depends only on the strategy, s, played by her and the multiset of strategies, S, played by the others, without reference to their identities. Furthermore, if any other player j were to play s and the remaining players S, the payoff to j would be identical to that of i. A *symmetric Nash equilibrium* (SNE) is a NE in which all players play the same strategy. Nash [15], while providing game theory with its central solution concept, also defined the notion of a symmetric game and proved, in a separate theorem, that such games always admit a symmetric equilibrium.

A simple reduction is known from 2-Nash to symmetric 2-Nash, and it shows that the latter is also PPAD-complete. The questions studied by Gilboa and Zemel [10] for 2-player games were studied by Conitzer and Sandholm [5] for symmetric games and were shown to be NP-complete. On the other hand, no reduction is known from 3-Nash to symmetric 3-Nash. Indeed, after giving the reduction from 2-Nash to symmetric 2-Nash, Papadimitriou [18] states, "Amazingly, it is not clear how to generalize this proof for three player games!"

Our second set of results deals with symmetric k-player games, for $k \geq 3$. We first give a reduction from 3-Nash to symmetric 3-Nash, hence settling the open problem of [18]. This also enables us to show that symmetric 3-Nash is complete for the class FIXP_a, Strong Approximation FIXP, which is a variant of FIXP that is meant for the Turing machine model. It also yields ETR-completeness for **Superset** and **Subset** in such games. Once the 3-player case is settled, we prove analogous results for symmetric k-player games, for $k > 3$.

[9] gave a dichotomy for NE, showing a qualitative difference between 2-Nash and k-Nash along three different criteria, see Table 1. The results of this paper add a fourth criterion to this dichotomy, namely complexity of decision problems. Additionally, we get an analogous dichotomy for symmetric NE, see Table 2. Results of current paper are indicated by $\mathcal{CP}$ in the tables.

Table 1.

	2-Nash	**k-Nash, $k \geq 3$**
Nature of solution	Rational [12]	Algebraic; irrational example [15]
Complexity	PPAD-complete [16,6,4]	FIXP-complete [7]
Practical algorithms	Lemke-Howson [12]	?
Decision problems	NP-complete [10,5]	ETR-complete: [22] $\mathcal{CP}$ (Theorems 13, 14)

Table 2.

	Symmetric 2-Nash	**Symmetric k-Nash, $k \geq 3$**
Nature of solution	Rational [12]	Algebraic; irrational example $\mathcal{CP}$ together with [15]
Complexity	PPAD-complete [16,6,4]	FIXP_a-complete: $\mathcal{CP}$ (Theorem 24)
Practical algorithms	Lemke-Howson [12,21]	?
Decision problems	NP-complete [5]	ETR-complete: $\mathcal{CP}$ (Theorems 23)

1.1 Technical Overview

We first give the main idea behind our reduction from 3-Nash to symmetric 3-Nash (Theorem 19). We will reduce the given game (A, B, C), where each tensor is $m \times n \times p$, to a symmetric game, D, of dimension $l \times l \times l$, where $l = m + n + p$ (see Section 2.1 for the description of (symmetric) games). In this game, under each symmetric NE, the strategy of each player can be decomposed into three vectors, say $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, of dimension m, n, p, respectively. An essential condition for recovering a Nash equilibrium for the original game (A, B, C) is that each of these three vectors be non-zero; this is also the most difficult part of the reduction.

To achieve this we construct a $3 \times 3 \times 3$ symmetric game G all of whose symmetric NE are of full support, even though it is only partially specified (see (4)). We "blow up" G to derive D, which is $l \times l \times l$, and the unspecified entries of G create room where tensors A, B, C are "inserted". Now, if $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a symmetric NE of D then so is $(\sum_i x_i, \sum_j y_j, \sum_k z_k)$ of G. As a result, each vector, $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \neq 0$. Next we show that if these vectors are scaled to probability vectors, they form a NE for (A, B, C). Additional arguments yield ETR-completeness for **Subset** and **Superset** for symmetric k-Nash (Theorems 20 and 21).

Next we give idea for showing that symmetric 3-Nash is complete for the class FIXP_a (Theorem 22), Strong Approximation FIXP, which is a variant of FIXP that is meant for the Turing machine model. Note that we are unable to show that symmetric 3-Nash is complete for the class FIXP itself, since we don't see how to express the solution to the given instance as a rational linear projection of the solution of the reduced instance.

Under FIXP_a, given an instance I and a rational $\epsilon > 0$, we need to compute a vector $\boldsymbol{x}$ that is within (additive) ϵ distance from some solution, i.e., $\exists \boldsymbol{x}^* \in Sol(I)$ such that $|\boldsymbol{x}^* - \boldsymbol{x}|_\infty \leq \epsilon$, in time polynomial in $size[I]$ and $\log(1/\epsilon)$. In the above reduction, obtaining a solution of (A, B, C) involves e.g., dividing $\boldsymbol{x}$ by $\sum_i x_i$. If the latter is very small, this may give us a vector that is very far away from a solution of (A, B, C), even though x may be close to a solution of D.

We get around this problem by a small change in the above reduction, namely, we need to multiply the tensors A, B, C by a small constant ϵ' before they are "inserted" at the appropriate places in G' to get symmetric game D. This ensures that $(\sum_i x_i, \sum_j y_j, \sum_k z_k)$ is approximately $(1/3, 1/3, 1/3)$. As a result, given a point close to a solution of D, we can get a point "close" to a solution of (A, B, C).

Next, we describe how we show ETR-completeness for the four decision problems, mentioned in the previous section, for k-Nash. To show hardness in case of 3-players, we reduce **InBox**, which is known to be ETR-complete for 3-Nash [22], to each of **MaxPayoff**, **Subset** and **Superset**, and then from **MaxPayoff** to **NonUnique**. Hardness for the k-Nash, $k > 3$, follows since 3-Nash reduces to k-Nash trivially by introducing dummy players. To show containment in ETR we give a Non-linear complementarity problem (NCP) formulation that exactly captures NE of a given game (Theorems 25 and 26).

Next, we briefly explain the reduction from **InBox** to **MaxPayoff** for the 2-player case (see Section 3.1 for details); 3-player case is an extension of it (Appendix C.1). Let the given game be represented by two payoff matrices (A, B) of dimension $m \times n$, one for each player. The **InBox** problem is to check if it has a NE in which all strategies are played with at most 0.5 probability. We reduce it to checking if another game (C, D) has a NE in which every player gets payoff at least $h > 0$ (**MaxPayoff**). Wlog we can assume that $A, B > 0$.

We construct $m(n+1) \times n(m+1)$ matrices C and D, where the top-left block is set to $A + h$ and $B + h$ respectively (see Figure 1). This ensures that if each player gets payoff h at a NE, then strategies from this block are played with non-zero probability, and normalizing them gives a NE of (A, B). The latter follows since NE set remains invariant under additive scaling of payoffs. In order to retrieve a NE in 0.5 ball, we ensure that if any of these strategies is (relatively) played with more than 0.5 probability then a sequence of deviations leads to both players playing only among their last mn strategies where payoff is zero $(< h)$.

In particular suppose the second player plays $\boldsymbol{y}$ in the top-left block. The last mn strategies of the row player are divided in to n blocks of size m, one for each $y_j,\ j \leq n$ such that if $y_j > 0.5$ then best response of the first player is to deviate to j^{th} block. The payoff of the second player is set to -1 in these blocks, so then y_j fetches -1 and second player is forced to deviate to her last mn strategies where both get zero. Similarly for the first player.

Due to space constraints, in next few pages we present overview of our two results (i) ETR-hardness for **MaxPayoff**, **Subset** and **Superset**, through reduction from **InBox**, (ii) reduction from 3-Nash to symmetric 3-Nash, and ETR-hardness of **Subset** and **Superset** for symmetric 3-Nash.

2 Preliminaries

In this section we formally define the (symmetric) k-Nash problem, and their decision problems. Further, we discuss the complexity classes ETR and FIXP.

Notations: All vectors are in bold-face letters, and i^{th} coordinate of vector $\boldsymbol{x}$ is denoted by x_i, and $\boldsymbol{x}^{-i}$ denotes the vector $\boldsymbol{x}$ with i^{th} coordinate removed. $\mathbf{1}$ and $\mathbf{0}$ represent all ones and all zeros vector respectively of appropriate dimension. For integers $k < l$, $\boldsymbol{x}(k:l) = (x_k, x_{k+1}, \ldots, x_l)$. We use $[n]$ to denote set $\{1, \ldots, n\}$ and $[k:l]$ to denote $\{k, k+1, \ldots, l\}$. If $\boldsymbol{x}$ is of m dimension, then by $\sigma(\boldsymbol{x})$ we mean $\sum_{i=1}^{m} x_i$, and $\eta(\boldsymbol{x}) = \boldsymbol{x}/\sigma(\boldsymbol{x})$. Concatenation of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by $(\boldsymbol{x}|\boldsymbol{y})$. Given a matrix A and $h \in \mathbb{R}$, $A+h$ denotes the matrix A with h added to each of its entries. Further, $A(i,:)$ is its i^{th} row and $A(:,j)$ is its j^{th} column.

2.1 (Symmetric) k-Nash

For a given k-player game let $S_i, i \in [k]$ be the set of pure strategies of player i, and let $\boldsymbol{S} = \times_i S_i$. The payoffs of player i can be represented by a k-dimensional tensor A_i, such that $A_i(\boldsymbol{s})$ denotes the payoff she gets when $\boldsymbol{s} \in \boldsymbol{S}$ is played. Players may randomize among their strategies. Let Δ_i denote the set of mixed strategy profiles of player i, and let $\boldsymbol{\Delta} = \times_i \Delta_i$. Expected payoff of player i from $\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k) \in \boldsymbol{\Delta}$ is $\pi_i(\boldsymbol{x}) = \sum_{\boldsymbol{s} \in \boldsymbol{S}} (\Pi_{i \in [k]} \boldsymbol{x}^i_{s_i}) A_i(\boldsymbol{s})$.

Definition 1. *(Nash Equilibrium (NE) [15])* $\boldsymbol{x} \in \boldsymbol{\Delta}$ *is said to be a NE if no player gains by unilateral deviation. Formally,* $\forall i,\ \forall \boldsymbol{x}' \in \Delta_i,\ \pi_i(\boldsymbol{x}) \geq \pi_i(\boldsymbol{x}', \boldsymbol{x}^{-i})$.

Let $\pi_i(s, \boldsymbol{x}^{-i})$ denote the payoff i receives when she plays $s \in S_i$ and others play as per $\boldsymbol{x}^{-i}$. It is easy to see that, $\boldsymbol{x}$ is a NE iff [15],

$$\forall i \in [k], \quad \forall s \in S_i, \quad x^i_s > 0 \quad \Rightarrow \quad \pi_i(s, \boldsymbol{x}^{-i}) = \max_{t \in S_i} \pi_i(t, \boldsymbol{x}^{-i}) \tag{1}$$

Symmetric k-Nash: In a symmetric game the players are indistinguishable. Their strategy sets are identical (S) and payoffs are symmetric represented by one tensor A. For a player, the payoff she gets by playing $s' \in S$, when others are playing $\boldsymbol{s} \in S^{k-1}$, is $A(s', \boldsymbol{s})$. Further, who is playing what in $\boldsymbol{s}$ does not matter. Formally, A satisfies $A(s', \boldsymbol{s}) = A(s', \boldsymbol{s}_\tau)$ for all permutations τ of $(1, \ldots, k-1)$, where $\boldsymbol{s}_\tau$ is the corresponding permuted vector.

A profile $\boldsymbol{x} \in \boldsymbol{\Delta}$ is called *symmetric* if $\boldsymbol{x}^i = \boldsymbol{x}^j,\ \forall i, j$, thus one vector $\boldsymbol{x} \in \Delta$ is enough to denote a symmetric profile. At a symmetric strategy profile all the players get the same payoff, and we denote it by $\pi(\boldsymbol{x})$. The problem of computing symmetric NE (SNE) of a symmetric game is called *symmetric k-Nash.*

Note that description of a (symmetric) k-player game takes $O(km^k)$ space, where $m = \max_i |S_i|$, which is exponential in m and k. To keep it polynomial, we consider k as a constant. Further, wlog $(A_1, \ldots, A_k) > 0$ because adding a constant to the tensors does not change the set of NE.

2-Nash: The payoff tensors in case of 2-player game are matrices, say (A, B), A for player one and B for player two. If the first player plays i and second plays j, then their respective payoff are A_{ij} and B_{ij}. Game is said to be symmetric if $B = A^T$. A mixed strategy is $(\boldsymbol{x}, \boldsymbol{y}) \in \Delta_1 \times \Delta_2$, and respective payoffs at such a strategy are $\boldsymbol{x}^T A\boldsymbol{y}$ and $\boldsymbol{x}^T B\boldsymbol{y}$. The NE characterization of (1) reduces to:

$$\forall i \in S_1,\ x_i > 0 \Rightarrow (A\boldsymbol{y})_i = \max_{k \in S_1}(A\boldsymbol{y})_k;\ \forall j \in S_2,\ y_j > 0 \Rightarrow (\boldsymbol{x}^T B)_j = \max_{k \in S_2}(\boldsymbol{x}^T B)_k \quad (2)$$

3-Nash: It is the k-Nash problem with 3 players. Such a game can be represented by 3-dimensional tensors (A, B, C); A for player one, B for player two, and C for player three. If player one plays i, two plays j and three plays k, then their respective payoffs are A_{ijk}, B_{ijk}, and C_{ijk}. If the game is symmetric then we have $A_{ijk} = A_{ikj} = B_{jik} = B_{kij} = C_{jki} = C_{kji}$. A mixed strategy is denoted by $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \Delta_1 \times \Delta_2 \times \Delta_3$. Thus NE characterization of (1) reduces to:

$$\begin{array}{lll} \forall i \in S_1,\ x_i > 0 & \Rightarrow & \sum_{j,k} A_{ijk} y_j z_k = \max_{l \in S_1} \sum_{j,k} A_{ljk} y_j z_k \\ \forall j \in S_2,\ y_j > 0 & \Rightarrow & \sum_{i,k} B_{ijk} x_i z_k = \max_{l \in S_2} \sum_{i,k} B_{ilk} x_i z_k \\ \forall k \in S_3,\ z_k > 0 & \Rightarrow & \sum_{i,j} C_{ijk} x_i y_j = \max_{l \in S_3} \sum_{i,j} C_{ijl} x_i y_j \end{array} \quad (3)$$

Decision Problems: Computational complexity of numerous decision problems have been studied for 2-Nash and 3-Nash [10,5]. Here are some interesting ones:

- **NonUnique**: If there exists more than one NE.
- **MaxPayoff**: Given a rational number h, if there exists a NE where every player gets payoff at least h.
- **Subset**: Given sets $T_i \subset \mathcal{S}_i,\ \forall i \in [1:k]$, if there exists a NE where every strategy in T_i is played with positive probability by player i.
- **Superset**: Given sets $T_i \subset \mathcal{S}_i,\ \forall i \in [1:k]$, if there exists a NE where all the strategies outside T_i are played with zero probability by player i.
- **InBox**: If there is a NE where every strategy is played with ≤ 0.5 probability.

All but last have been shown to be NP-complete in case of 2-Nash [10,5], and the last one is shown to be ETR-complete in case of 3-Nash [22]. In this paper, we show ETR-completeness for the first four decision problems for k-Nash, and for third and fourth for symmetric k-Nash.

2.2 Existential Theory of Reals (ETR)

In order to capture decision problems arising in *existential theory of reals* (ETR), Schaefer and Štefankovič [22] defined complexity class ETR as follows: An instance I of class ETR consists of a sentence of the form,

$$(\exists x_1, \ldots, x_n)\ \ \phi(x_1, \ldots, x_n),$$

where ϕ is a quantifier-free $(\wedge, \vee, \neg)$-Boolean formula over the predicates (sentences) defined by signature $\{0, 1, -1, +, *, <, \leq, =\}$ over variables that take real values. The question is if the sentence is true. The size of the problem is $n + size(\phi)$, where n is the number of variables and $size(\phi)$ is the minimum number of signatures needed to represent ϕ (we refer the reader to [22] for more details on ETR, and its relation with other classes like PSPACE). Schaefer and Štefankovič showed that for 3-Nash, problem **InBox** is ETR-complete.

2.3 The class FIXP and its variant FIXP_a

Etessami and Yannakakis [7] defined the class FIXP to capture complexity of the exact fixed point problems with algebraic solutions. A FIXP problem is to find a fixed-point of a function $F : D \to D$ over a convex, compact domain D. The function is given by an arithmetic circuit C with $\{\min, \max, +, -, *, /\}$ operations, rational constants, and n input/output. Size of C is $n + \#$ gates+ size(constants). Given $\boldsymbol{\lambda} \in D$ to C as an input, all its gates are well defined.

Fixed-points of F may be irrational. To remain faithful to Turing machine computation, Etessami and Yannakakis [7] defined a discrete class FIXP_a.

(Strong) Approximation FIXP_a: Given an instance I and a rational $\epsilon > 0$, compute a vector $\boldsymbol{x}$ that is within (additive) ϵ distance from some solution, i.e., $\exists \boldsymbol{x}^* \in Sol(I)$ such that $|\boldsymbol{x}^* - \boldsymbol{x}|_\infty \leq \epsilon$, in time polynomial in $size[I]$ and $\log(1/\epsilon)$.

Theorem 2. *[7] Given a 3-player game (A_1, A_2, A_3), computing its NE is FIXP-complete. The corresponding (Strong) Approximation is complete for FIXP_a.*

3 k-Nash: ETR-completeness for Decision Problems

In this section we show that **MaxPayoff**, **Subset**, **Superset** and **NonUnique** are ETR-hard in k-player games, for any constant $k \geq 3$; refer Appendix A for containment in ETR. It suffices to show the results for 3-Nash, as a 3-player game can be reduced to a k-player game trivially by adding $k-3$ dummy players, with one strategy each. To show hardness for **MaxPayoff**, **Subset** and **Superset** we reduce from **InBox**, and for **NonUnique** we reduce from **MaxPayoff**.

3.1 InBox to MaxPayoff, Subset and Superset

To convey the main ideas, we first describe the reduction in 2-player games and later generalize it to the 3-player case (in Appendix C). We show the reduction from **InBox** to **MaxPayoff**, and from the intermediate lemmas, reduction to **Subset** and **Superset** will follow. Let the given two player game be represented by $m \times n$ dimensional payoff matrices $(A, B) > 0$.

For $a \geq 0$, let $\mathcal{B}_a = [0,\ a]^{m+n}$ be a ball of radius a at origin in l_∞ norm. We will construct another game (C, D), with $m(n+1) \times n(m+1)$-dimensional matrices, and show that it has a NE where each player gets at least $h > 0$ payoff (**MaxPayoff**) if and only if the game (A, B) has a NE in ball $\mathcal{B}_{0.5}$ (**InBox**). First we define a couple of notations required for the construction.

Definition 3. *Let i and j be integers where $i \in [m]$ and $j \in [n]$, and h be a real number. We define the following operators:*

$A_{(i,:)+h}$ *: matrix A with h added to the entries in its i^{th} row.*

$A_{(:,j)+h}$ *: matrix A with h added to the entries in its j^{th} column.*

Definition 4. *Given a matrix M of dimension $a \times b$ and integers r, s such that $a + r - 1 \le m(n+1)$ and $b + s - 1 \le n(m+1)$, define $[M]_{r,s}$ to be an $m(n+1) \times n(m+1)$-dimensional matrix where M is copied starting at position (r, s), and all other coordinates are set to zero.*

Using the above notations we construct matrices C, D as follows, where $h > 0$ (see Figure 1 in Appendix B):

$$C = [A + h]_{1,1} + [(-1)_{m \times mn}]_{1,n+1} + \textstyle\sum_{j \in [n]} [A_{(:,j)+2h}]_{jm+1,1}$$
$$D = [B + h]_{1,1} + [(-1)_{mn \times n}]_{m+1,1} + \textstyle\sum_{i \in [m]} [B_{(i,:)+2h}]_{1,in+1}$$

The next lemma follows from the construction of C, D. Recall $\sigma(\boldsymbol{x}) = \sum_i x_i$.

Lemma 5. *Given a strategy $(\boldsymbol{x}', \boldsymbol{y}')$ of game (C, D), let $\boldsymbol{x} = \boldsymbol{x}'(1 : m)$, $\boldsymbol{y} = \boldsymbol{y}'(1 : n)$, $\alpha = h * \sigma(\boldsymbol{y}) - \sigma(\boldsymbol{y}'(n+1 : (m+1)n))$, and $\beta = h * \sigma(\boldsymbol{x}) - \sigma(\boldsymbol{x}'(m+1 : (n+1)m))$. Then,*

$$(C\boldsymbol{y}')_i = \begin{cases} \alpha + (A\boldsymbol{y})_i & \text{if } i \in [m] \\ 2hy_{\lfloor (i-1)/m \rfloor} + (A\boldsymbol{y})_r & \text{if } i \in [m+1, m(n+1)],\ r = ((i-1) \bmod m) + 1. \end{cases}$$
$$(\boldsymbol{x}'^T D)_j = \begin{cases} \beta + (\boldsymbol{x}^T B)_j & \text{if } j \in [n] \\ 2hx_{\lfloor (j-1)/n \rfloor} + (\boldsymbol{x}^T B)_r & \text{if } j \in [n+1, n(m+1)], r = ((j-1) \bmod n) + 1. \end{cases}$$

Before the formal reduction, here is a brief intuition. Note that in (C, D) we have copied $(A + h, B + h)$ in the top-left $m \times n$ block, we call it *first block* now on. Since adding a constant does not change NE of a game, if strategies from the first block are played with non-zero probability at a NE of (C, D), then it may give a NE of (A, B). This is ensured if payoffs achieved at the NE are positive (or at least $h > 0$; a solution of **MaxPayoff**), using Lemma 5.

To guarantee a in $\mathcal{B}_{0.5}$ for game (A, B) (solution of **InBox**), we make use of the blocks added after the first block in both the directions. In particular, in Lemma 5, if $\exists j \in [n],\ y_j > 0.5 * \sigma(\boldsymbol{y})$, then for the first player her first m strategies are worse than those from block $[mj + 1 : mj + m]$, forcing her to play only from her last mn strategies. This will force the second player to move away from the first block too (or else he gets $-ve$ payoff), and thereby leading to a NE where both play from last mn strategies and both get zero payoff (not a solution of **MaxPayoff**). We will use these observations crucially in the reduction.

For game (A, B) only those NE $(\boldsymbol{x}, \boldsymbol{y})$ are interesting which satisfy $\boldsymbol{x}, \boldsymbol{y} \le 0.5$ (solutions of **InBox**). We show that such NE are retained as NE of (C, D). The proof uses the fact that in C and D, top-left block encodes A and B respectively.

Lemma 6. *(A, B) has a NE $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{B}_{0.5}$ iff $((\boldsymbol{x}, 0_{mn}), (\boldsymbol{y}, 0_{mn}))$ is a NE of (C, D).*

Lemma 6 maps a solution of **InBox** in game (A, B) to a NE of (C, D) where players play only among their first m, n strategies respectively. Next we show a reverse mapping: a NE of (C, D) where both players play some of first m, n strategies, gives a NE of game (A, B). Recall that for vector $\boldsymbol{x}$, $\eta(\boldsymbol{x}) = {}^{\boldsymbol{x}}/_{\sigma(\boldsymbol{x})}$.

Lemma 7. *If* $(\boldsymbol{x}', \boldsymbol{y}')$ *is a NE of game* (C, D) *s.t.* $\boldsymbol{x} = \boldsymbol{x}'[1:m]$ *and* $\boldsymbol{y} = \boldsymbol{y}'[1:n]$ *are non zero, then* $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}))$ *is a NE for game* (A, B)*, and* $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y})) \in \mathcal{B}_{0.5}$*.*

Lemmas 6 and 7 implies that game (A, B) has a NE in $\mathcal{B}_{0.5}$ if and only if game (C, D) has a NE where both the players play some of first m, n strategies respectively. If we show that to get payoff of at least h in the latter game, players have to play some of first m, n strategies, then clearly the reduction will follow.

Lemma 8. *Given a strategy profile* $(\boldsymbol{x}', \boldsymbol{y}')$*, if* $\boldsymbol{x}'^T C \boldsymbol{y}' \geq h$ *and* $\boldsymbol{x}'^T D \boldsymbol{y}' \geq h$ *then* $\boldsymbol{x} = \boldsymbol{x}'(1:m)$ *and* $\boldsymbol{y} = \boldsymbol{y}'(1:n)$ *are non-zero.*

The next theorem follows using Lemmas 6, 7, and 8.

Theorem 9. *Game* (A, B) *has a NE in ball* $\mathcal{B}_{0.5}$ *if and only if game* (C, D) *has a NE where every player gets payoff at least* h*.*

Next theorem shows reduction from **InBox** to **Superset** using Lemma 6.

Theorem 10. *Game* (A, B) *has a NE in* $\mathcal{B}_{0.5}$ *if and only if game* (C, D) *has a NE where all the strategies played with non-zero probability by first and second player are from* $T_1 = [1:m]$ *and* $T_2 = [1:n]$*.*

Lemmas 6 and 7 imply that, one of first m, n strategies are played with non-zero probability by respective players in game (C, D) if and only if game (A, B) has a NE in ball $\mathcal{B}_{0.5}$. Thus next theorem gives a Turing (and not a many-one) reduction from **InBox** to **Subset**.

Theorem 11. *Game* (A, B) *has a NE in ball* $\mathcal{B}_{0.5}$ *if and only if* $\exists i \in [m], \exists j \in [n]$ *such that for* $T_1 = \{i\}$ *and* $T_2 = \{j\}$*, game* (C, D) *has a NE where all strategies of* T_1 *and* T_2 *are played with non-zero probability.*

In Appendix C we extend Theorems 9, 10 and 11 to 3-player games in Theorems 33, 34 and 35 respectively. These together with ETR-hardness of **InBox** in 3-Nash [22], and Theorem 25 showing containment in ETR gives the next result.

Theorem 12. *Problems* **MaxPayoff***,* **Subset** *and* **Superset** *are ETR-complete in* 3*-player games.*

A 3-player game can be reduced to a k-player game trivially, without changing its set of NE, by adding $k-3$ dummy players with one strategy each (and payoff tensor $A_i = [h]$ to get reduction for **MaxPayoff**). And therefore, the next theorem follows from Theorems 12 and 25.

Theorem 13. *Given a* k*-player game* $(A_1, \ldots, A_k)$*, for a constant* $k \geq 3$*, problems of* **NonUnique***,* **MaxPayoff***,* **Subset** *and* **Superset** *are ETR-complete.*

Finally, to show ETR-completeness for **NonUnique**, in Appendix C.1 we reduce **MaxPayoff** to **NonUnique** in 3-player games (Theorem 42), and thereby obtain (using Theorem 25),

Theorem 14. *Given a* k*-player game* $(A_1, \ldots, A_k)$*, for a constant* $k \geq 3$*, problem of* **NonUnique** *is ETR-complete.*

4 Symmetric 3-Nash: ETR and $\text{FIXP}_{\boldsymbol{a}}$ Completeness

In this section, we give a reduction from 3-Nash to symmetric 3-Nash, and thereby obtain ETR-hardness for **Subset** and **Superset**, and FIXP_a-hardness; for containment in ETR and FIXP_a see Appendices A and D respectively.

Let the given game be (A, B, C), where each tensor is $m \times n \times p$. Let D denote the reduced symmetric game, which will be of dimension $l \times l \times l$, where $l = m + n + p$. Let $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ be a NE of (A, B, C). We will show that there are positive numbers α, β, γ such that $(\boldsymbol{d}, \boldsymbol{d}, \boldsymbol{d})$ is a NE of the reduced game, where $\boldsymbol{d}$ is a l-dimensional vector $(\alpha\boldsymbol{x}|\beta\boldsymbol{y}|\gamma\boldsymbol{z})$. Furthermore, let $(\boldsymbol{d}, \boldsymbol{d}, \boldsymbol{d})$ be a NE of the reduced game, where $\boldsymbol{d}$ decomposes into vectors $\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}'$ of dimension m, n, p respectively. Scaling these vectors gives a NE $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ of game (A, B, C). This will yield mapping in both directions.

Essential to this reduction is the $3 \times 3 \times 3$ symmetric game G given below. We represent the payoff tensor of the first player by three 3×3 matrices, one for each of her pure strategy. Here a, b, c are any non-negative reals.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & a \\ 0 & a & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

Lemma 15. *If (α, β, γ) is a symmetric NE of game G, then $\alpha, \beta, \gamma > 0$.*

From G, we derive symmetric game D, which is $l \times l \times l$, by blowing up each of the three strategies of G to m, n, p number of strategies respectively. Copy 0s and 1s to their respective blocks, and replace blocks corresponding to a, b, c by A, B, C respectively. For a formal description of D see (8) in Appendix E.

In the above game, suppose two players are playing mixed-strategy $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$, where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are of dimensions m, n, p respectively. Then from strategy s the third player receives payoff:

$$\pi^D(s, \boldsymbol{d}) = \begin{cases} (\sigma(\boldsymbol{y}))^2 + 2\sum_{j\in[n],k\in[p]} A_{sjk} y_j z_k, & \text{if } s \le m, \\ (\sigma(\boldsymbol{z}))^2 + 2\sum_{i\in[m],k\in[p]} B_{isk} x_i z_k & \text{if } m < s \le m+n \\ (\sigma(\boldsymbol{x}))^2 + 2\sum_{i\in[m],j\in[n]} C_{ijs} x_i y_j & \text{if } m+n < s \le l \end{cases} \tag{5}$$

Wlog we assume that $A, B, C \ge 0$ and hence $D \ge 0$. We consider $\frac{0}{0}$ as 0.

Lemma 16. *If $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$ is a SNE of game D then $(\sigma(\boldsymbol{x}), \sigma(\boldsymbol{y}), \sigma(\boldsymbol{z}))$ is a NE of G where $a = \frac{\max_{s\le m}\sum_{jk} A_{sjk} y_j z_k}{\sigma(\boldsymbol{y})\sigma(\boldsymbol{z})}$, $b = \frac{\max_{s\le n}\sum_{i,k} B_{isk} x_i z_k}{\sigma(\boldsymbol{x})\sigma(\boldsymbol{z})}$, $c = \frac{\max_{s\le p}\sum_{i,j} C_{ijs} x_i y_j}{\sigma(\boldsymbol{x})\sigma(\boldsymbol{y})}$.*

Lemmas 15 and 16 imply that at any SNE $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$, all three components $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ of the strategy profile are non-zero. Next we show that normalizing each gives a NE of the original game (A, B, C).

Lemma 17. *If $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$ is a SNE of game D, then $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(\boldsymbol{z}))$ is a NE of game (A, B, C).*

The mapping from SNE of game D to NE of game (A,B,C) established in Lemma 17 implies that computing SNE in symmetric games is no easier than computing a NE in normal games. We extend this reduction to k-Nash in Appendix G. Next, we show a mapping in reverse direction, i.e., from NE of (A,B,C) to a SNE of D, to obtain ETR-hardness results for a number of decision problems in symmetric 3-Nash.

Lemma 18. *Let $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ be a NE of (A,B,C), and let (α,β,γ) be a NE of game G where a,b,c are set to payoffs of the first, second and third players respectively at the NE of game (A,B,C). Then $\boldsymbol{d}=(\alpha\boldsymbol{x}|\beta\boldsymbol{y}|\gamma\boldsymbol{z})$ is a SNE of game D.*

The next theorem summaries the relation between NE of game (A,B,C) and SNE of game D, and follows using Lemmas 17 and 18.

Theorem 19. *Profile $\boldsymbol{d}=(\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$ is a SNE of game D iff $(\eta(\boldsymbol{x}),\eta(\boldsymbol{y}),\eta(\boldsymbol{z}))$ is a NE of game (A,B,C).*

We showed a number of ETR-completeness results for 3-Nash in Section 3. Since, support of NE remains intact in the reduction from 3-Nash to symmetric 3-Nash as shown in Theorem 19, next we show ETR-completeness of **Subset** and **Superset** problems for symmetric 3-Nash.

Theorem 20. *Given a symmetric game D and a subset $T\subset S$, it is ETR-complete to check if there exists a SNE $\boldsymbol{x}$ s.t. $x_s>0,\ \forall s\in T$ (***Subset***).*

The next theorem follows similarly using Theorems 12 and 19.

Theorem 21. *Given a symmetric game D and a subset $T\subset S$, it is ETR-complete to check if there exists a SNE $\boldsymbol{x}$ s.t. $x_s=0,\ \forall s\in S\setminus T$ (***Superset***).*

Even though Theorem 19 reduces 3-Nash, which is known to be FIXP-complete [7], to symmetric 3-Nash, we do not get FIXP-harness for the latter. This is because to obtain a solution, say $\boldsymbol{x}$, of former requires *division* among the coordinates of a solution, say $\boldsymbol{d}$, of latter. While FIXP reduction requires that every x_i is a linear function of some d_j, with rational coefficients (because of irrational solutions). Instead, in Appendix F we show FIXP_a-completeness for symmetric 3-Nash which always has a rational solution, and obtain the following.

Theorem 22. *Symmetric 3-Nash is FIXP_a-complete.*

Since there is no trivial extension of symmetric 3-player game to symmetric k-player game, in Appendix G we extend Theorems 20, 21 and 22 to symmetric k-Nash, and show the following.

Theorem 23. *For symmetric k-Nash, problems* **Subset** *and* **Superset** *are ETR-complete, where $k\geq 3$ is a constant.*

Theorem 24. *For a constant $k\geq 3$, symmetric k-Nash is FIXP_a-complete.*

We refer the reader to Appendix H for a discussion on the significance of our results and open questions.

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A (Symmetric) k-Nash: Containment in ETR

In this section we show that the first four decision problems described in Section 2.1 are in ETR, for k-Nash as well as symmetric k-Nash. For a k-player game $(A_1, \ldots, A_k)$, NE characterization of (1) can be reformulated as a set of polynomial inequalities as follows, where variable x_s^i captures the probability with which player i plays $s \in S_i$, and variable λ_i captures her best payoff.

$$\forall i \in [1:k], \quad \forall s \in S_i, \quad x_s^i \geq 0; \quad \pi_i(s, \boldsymbol{x}^{-i}) \leq \lambda_i; \quad x_s^i(\pi_i(s, \boldsymbol{x}^{-i}) - \lambda_i) = 0 \quad (6)$$

It is easy to see that strategy profile $\boldsymbol{x} \in \boldsymbol{\Delta}$ satisfies (1) if and only if it satisfies (6).

Theorem 25. *Given a k-player game $(A_1, \ldots, A_k)$, for a constant k, the problems of* **NonUnique***,* **MaxPayoff***,* **Subset** *and* **Superset** *are in ETR.*

Proof. To frame **NonUnique** as an ETR problem, take two copies of (6) each with different sets of variables, say $\boldsymbol{x}$ and $\boldsymbol{y}$, and add $|\boldsymbol{x} - \boldsymbol{y}|^2 > 0$ to it. This system has a feasible solution $(\boldsymbol{x}, \boldsymbol{y})$ if and only if the game has two NE $\boldsymbol{x} \neq \boldsymbol{y}$. Thus, containment of **NonUnique** in ETR follows.

For **MaxPayoff**, add $\forall i \in [1:k], \ \pi_i(\boldsymbol{x}) \geq h$ to the system (6). It has a feasible solution $\boldsymbol{x}$ if and only if $\boldsymbol{x}$ is a NE of the game where payoff received by every player is at least h, implying **MaxPayoff** is in ETR.

Similarly, to formulate **Subset**, add $\forall i \in [1:k], \ \forall s \in T_i, \ x_s^i > 0$ to (6). And for **Superset**, add $\forall i \in [1:k], \ \forall s \in S_i \setminus T_i, \ x_s^i = 0$ to (6). ☐

Given a symmetric game A, the following system of polynomial inequalities (similar to (6)) exactly captures its symmetric NE, where variable x_s captures the probability of playing strategy $s \in S$ and λ captures the payoff.

$$\forall s \in S, \quad x_s \geq 0; \quad \pi(s, \boldsymbol{x}) \leq \lambda; \quad x_s(\pi(s, \boldsymbol{x}) - \lambda) = 0$$

The proof for the next theorem follows similar to that of Theorem 25.

Theorem 26. *Given a symmetric k-player game A, for a constant k, the problems of* **NonUnique***,* **MaxPayoff***,* **Subset** *and* **Superset** *for symmetric NE are in ETR.*

B Missing proofs of section 3

Proof of Lemma 6. To prove forward direction, it suffices to check if $(\boldsymbol{x}', \boldsymbol{y}')$ satisfies (2) for game (C, D). We show the conditions for the first player, namely involving C, and proof for the second player follows similarly. As last mn strategies in $\boldsymbol{y}'$ are not played at all, we have $\alpha = h \sum_{j \in [n]} y_i - \sum_{j \in [n+1, n(m+1)]} y'_j = h * 1 - 0 = h$. This together with Lemma 5 gives,

$$i \in [m], \ (C\boldsymbol{y}')_i = h + (A\boldsymbol{y})_i \quad \Rightarrow \quad \max_{i \in [m]} (C\boldsymbol{y}')_i = h + \max_{i \in [m]} (A\boldsymbol{y})_i$$

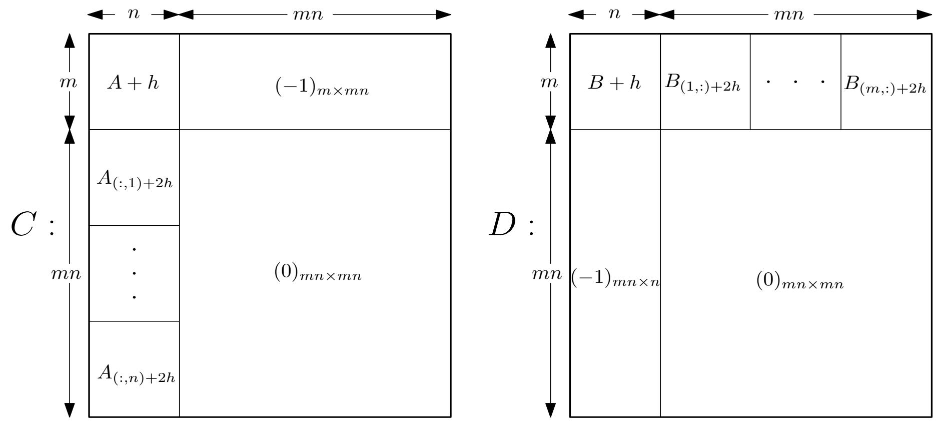

Fig. 1.

For $i \in [m+1, m(n+1)]$, let $r = ((i-1) \mod m) + 1$ and $k = \lfloor (i-1)/m \rfloor$. Then using Lemma 5 and the fact that $y_k \leq 0.5$, we have

$$(C\boldsymbol{y}')_i \leq 2h(0.5) + (A\boldsymbol{y})_r = h + (A\boldsymbol{y})_r = (C\boldsymbol{y}')_r$$

In other words strategies $[1:m]$ give at least as much payoff as the rest. Since $(\boldsymbol{x}, \boldsymbol{y})$ is a NE of game (A, B), if $x'_i = x_i > 0$ then $(C\boldsymbol{y}')_i = h + (A\boldsymbol{y})_i = h + \max_{k\in[m]}(A\boldsymbol{y})_k = \max_{k\in[m(n+1)]}(C\boldsymbol{y}')_k$.

For the reverse direction, $\exists i \in [m] x'_i > 0$ and hence $\forall j \in [n],\ (C\boldsymbol{y}')_i \geq (C\boldsymbol{y}')_{mj+i} \Rightarrow 2hy_j \leq h \Rightarrow y_j \leq 0.5$. Similarly $\boldsymbol{x} \leq 0.5$ follows. □

Proof of Lemma 7. As $\sigma(\boldsymbol{x}), \sigma(\boldsymbol{y}) > 0$, to show $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y})$ is a NE of (A, B) it suffices to show the following.

$$\forall i \in [m], x_i > 0 \ \Rightarrow \ (A\boldsymbol{y})_i = max_{k\in[m]}(A\boldsymbol{y})_k$$
$$\forall j \in [n], y_j > 0 \ \Rightarrow \ (\boldsymbol{x}^T B)_j = max_{k\in[n]}(\boldsymbol{x}^T B)_k$$

We show that the first one holds, and the proof for the second follows similarly. Let $\lambda = max_{k\in[m(n+1)]}(C\boldsymbol{y}')_k$ and $\lambda' = max_{k\in[m]}(C\boldsymbol{y}')_k = \alpha + \max_{k\in[m]}(A\boldsymbol{y})_k$ (Using Lemma 5). As $\exists i \in [m], x'_i > 0$ we have $\lambda' = \lambda$. Thus we get,

$$\forall i \in [m], x_i > 0 \Rightarrow (C\boldsymbol{y}')_i = \lambda \Rightarrow \alpha + (A\boldsymbol{y})_i = \alpha + \max_{k\in[m]}(A\boldsymbol{y})_k \Rightarrow (A\boldsymbol{y})_i = \max_{k\in[m]}(A\boldsymbol{y})_k$$

For the second part, to the contrary suppose $\exists j \in [n],\ (\eta(\boldsymbol{y}))_j = \frac{y_j}{\sigma(\boldsymbol{y})} > 0.5 \Rightarrow 2y_j > \sigma(\boldsymbol{y})$. Then for some $i \in [m]$ we have $x'_i > 0$ and $(C\boldsymbol{y}')_i \leq h\sigma(\boldsymbol{y}) + (A\boldsymbol{y})_i < 2hy_j + (A\boldsymbol{y})_i = (C\boldsymbol{y}')_{jm+i} \leq \lambda$, a contradiction to $(\boldsymbol{x}', \boldsymbol{y}')$ being a NE of game (C, D). □

Proof of Lemma 8. If $\boldsymbol{y} = \boldsymbol{0}$, then $\forall i \in [m(n+1)]$ we have $(C\boldsymbol{y}')_i \leq 0$ using Lemma 5, and in turn $\boldsymbol{x}'^T C\boldsymbol{y}' \leq 0$. Similarly, if $\boldsymbol{x} = 0$, then $\forall j \in [n(m+1)$

we have $(\boldsymbol{x}'^T D)_j \leq 0$, and then $\boldsymbol{x}'^T D \boldsymbol{y}' \leq 0$. Lemma follows using the fact that $h > 0$. □

C 3-Nash: InBox to MaxPayoff, Subset and Superset

Like in the two player case, given a 3-player game with $m \times n \times p$-dimensional payoff tensors (A, B, C), we will create a game (D, E, F) of dimension $m(n+1) \times n(p+1) \times p(m+1)$ and insert the original game in the first block with h added. We start with the definitions, analogous to those in Section 3.1.

Definition 27. *For $i \in [m]$, $j \in [n]$, $k \in [p]$, and a real number h, define*
$A_{(i,:,:)+h}$ *: Tensor A with h added to the entries $a_{ij'k'}$ $\forall j' \in [n], \forall k' \in [p]$.*
$A_{(:,j,:)+h}$ *: Tensor A with h added to the entries $a_{i'jk'}$ $\forall i' \in [m], \forall k' \in [p]$.*
$A_{(:,:,k)+h}$ *: Tensor A with h added to the entries $a_{i'j'k}$ $\forall i' \in [m], \forall j' \in [n]$.*

Definition 28. *Given a tensor T of dimension $a \times b \times c$ and integers r, s, t s.t. $a+r-1 \leq m(n+1), b+s-1 \leq n(p+1)$ and $c+t-1 \leq p(m+1)$, define $[T]_{r,s,t}$ to be an $m(n+1) \times n(p+1) \times p(m+1)$ dimensional tensor where T is copied starting at position (r, s, t), and all other coordinates are set to zero.*

Construct game (D, E, F) as follows.

$$D = [A+h]_{1,1,1} + [(-1)_{m,n(p+1),mp}]_{1,1,p+1} + \sum_{j \in [n]} [A_{(:,j,:)+2h}]_{jm+1,1,1}$$

$$E = [B+h]_{1,1,1} + [(-1)_{mn,n,(m+1)p}]_{m+1,1,1} + \sum_{k \in [p]} [B_{(:,:,k)+2h}]_{1,kp+1,1}$$

$$F = [C+h]_{1,1,1} + [(-1)_{m(n+1),np,p}]_{1,n+1,1} + \sum_{i \in [m]} [C_{(i,:,:)+2h}]_{1,1,in+1}$$

Recall that $\pi_i(\boldsymbol{x})$, for $\boldsymbol{x} \in \boldsymbol{\Delta}$ represents the payoff of player i what played profile is $\boldsymbol{x}$. Since we will be dealing with two games in this section, in order to resolve ambiguity we super-script it with the payoff tensor under consideration, i.e., $\pi_1^A(\boldsymbol{d})$. To denote payoff from a pure-strategy i, when other two are playing $\boldsymbol{y}, \boldsymbol{z}$ we use $\pi_1^A(i, \boldsymbol{y}, \boldsymbol{z})$, even if $\boldsymbol{y}, \boldsymbol{z}$ are not probability distributions.

Lemma 29. *Let $\boldsymbol{y}'$ and $\boldsymbol{z}'$ be vectors of sizes $n(p+1)$ and $p(m+1)$. Let $\boldsymbol{y} = \boldsymbol{y}'[1:n]$, $\boldsymbol{z} = \boldsymbol{z}'[1:p]$ and $\alpha = h * \sigma(\boldsymbol{y})\sigma(\boldsymbol{z}) - \sum_{j \in [p+1, p(m+1)]} z'_j$. We have*

$$\pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \begin{cases} \alpha + \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) & \text{if } i \in [m] \\ 2hy_{\lfloor (i-1)/m \rfloor} + \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) & \text{if } i \in [m+1, m(n+1)], \\ & \quad \text{where } r = ((i-1) \bmod m) + 1 \end{cases}$$

Let $\mathcal{B}_{0.5} = [0, 0.5]^{m+n+p}$. Using the payoff structure in game (D, E, F) we show the next lemma.

Lemma 30. *Game (A, B, C) has a NE $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathcal{B}_{0.5}$ iff*

$$(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}') = ((\boldsymbol{x}, 0_{mn}), (\boldsymbol{y}, 0_{np}), (\boldsymbol{z}, 0_{mp}))$$

is a NE of the game (D, E, F).

Proof. The proof is similar to that of Lemma 6, For the forward direction, we show the first condition of (3) characterizing 3-Nash, and other two follow similarly. Again $\alpha = h$, and hence $\max i \in [m] \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = h + \max_{i \in [m]} \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z})$ (Using Lemma 29). Further, $\forall j \in [n]$ and $\forall i \in [m]$, we have $\pi_1^D(jm + i, \boldsymbol{y}', \boldsymbol{z}') = 2hy_j + \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) \leq h + \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) = \pi_i^D(i, \boldsymbol{y}', \boldsymbol{z}')$. Thus, first m strategies at least as good as last $[m+1, m(n+1)]$. We get $\forall i \in [m(n+1)] x_i' > 0 \Rightarrow \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \max_{s \in [m(n+1)]} \pi_1^D(s, \boldsymbol{y}', \boldsymbol{z}')$.

For the reverse direction, $\exists i \in [m] x_i' > 0$ and hence $\forall j \in [n],\ \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') \geq \pi_1^D(mj + i, \boldsymbol{y}', \boldsymbol{z}') \Rightarrow 2hy_j \leq h \Rightarrow y_j \leq 0.5$. Similarly $\boldsymbol{x} \leq 0.5$ and $\boldsymbol{z} \leq 0.5$ follows. □

Next we obtain a solution of **InBox** for game (A, B, C) from a NE of (D, E, F) with some special property.

Lemma 31. *If $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ is a NE of game (D, E, F) such that the vectors $\boldsymbol{x} = \boldsymbol{x}'[1 : m]$, $\boldsymbol{y} = \boldsymbol{y}'[1 : n]$, and $\boldsymbol{z} = \boldsymbol{z}'[1 : p]$ are non-zero, then $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(\boldsymbol{z}))$ is a NE for the game (A, B), and $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(\boldsymbol{z})) \in \mathcal{B}_{0.5}$.*

Proof. As $\sigma(\boldsymbol{x}), \sigma(\boldsymbol{y}), \sigma(\boldsymbol{z}) > 0$, profile $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(\boldsymbol{z})$ is well-defined. To show that it is NE of game (A, B, C) it suffices to show the following for the first player, and similar argument follows for the other two players.

$$\forall i \in [m],\ x_i > 0 \quad \Rightarrow \quad \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) = \max_{l \in [m]} \pi_1^A(l, \boldsymbol{y}, \boldsymbol{z})$$

Let $\lambda = \max_{k \in [m(n+1)]} \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}')$, and $\lambda' = \max_{k \in [m]} \pi_1^D(k, \boldsymbol{y}', \boldsymbol{z}') = \alpha + \max_{k \in [m]} \pi_1^A(k, \boldsymbol{y}, \boldsymbol{z})$ (Using Lemma 29). As $\exists i \in [m], x_i' > 0$ we have $\lambda' = \lambda$. Thus we get,

$$\begin{aligned}
\forall i \in [m], \quad x_i > 0 &\Rightarrow x_i' > 0 \\
&\Rightarrow \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \lambda \\
&\Rightarrow \alpha + \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) = \alpha + \max_{k \in [m]} \pi_1^A(k, \boldsymbol{y}', \boldsymbol{z}') \\
&\Rightarrow \pi_1^A(i, \boldsymbol{y}', \boldsymbol{z}') = \max_{k \in [m]} \pi_1^A(k, \boldsymbol{y}', \boldsymbol{z}')
\end{aligned}$$

For the second part, to the contrary suppose $\exists j \in [n],\ (\eta(\boldsymbol{y}))_j = \frac{y_j}{\sigma(\boldsymbol{y})} > 0.5 \Rightarrow 2y_j > \sigma(\boldsymbol{y})$. Then for some $i \in [m]$ we have $x_i' > 0$ and $\pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') \leq h\sigma(\boldsymbol{y}) + \pi_1^A(i, \boldsymbol{y}', \boldsymbol{z}') < 2hy_j + \pi_1^A(i, \boldsymbol{y}', \boldsymbol{z}') = \pi_1^D(jm + i, \boldsymbol{y}', \boldsymbol{z}') \leq \lambda$, a contradiction to $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ being a NE of game (D, E, F). □

Now if we can relate the NE of (D, E, F) where at least one of first m, n, p strategies are played by the respective players, and the payoff received at the NE by all the players, then **InBox** to **MaxPayoff** reduction will follow.

Lemma 32. *Given a strategy profile* $\boldsymbol{d} = (\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ *of game* (D, E, F)*, if* $\pi_i(\boldsymbol{d}) \geq h,\ i = 1, 2, 3$*, then* $\boldsymbol{x} = \boldsymbol{x}'(1:m)$*,* $\boldsymbol{y} = \boldsymbol{y}'(1:n)$ *and* $\boldsymbol{z} = \boldsymbol{z}'(1:p)$ *are non-zero.*

Proof. If $\boldsymbol{y} = \boldsymbol{0}$, then $\forall i \in [m(n+1)]$ we have $\pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') \leq 0$ using Lemma 29, and in turn $\pi_1(\boldsymbol{d}) \leq 0$. Similarly, if $\boldsymbol{z} = 0$ then we get $\pi_2(\boldsymbol{d}) \leq 0$, and if $\boldsymbol{x} = 0$ then $\pi_3(\boldsymbol{d}) \leq 0$. Lemma follows using the fact that $h > 0$. □

The next theorem, for **InBox** to **MaxPayoff** reduction, follows using Lemmas 30, 31, and 32.

Theorem 33. *Game* (A, B, C) *has a NE in ball* $\mathcal{B}_{0.5}$ *if and only if game* (D, E, F) *has a NE where every player gets payoff at least* h*.*

The next theorem showing reduction from **InBox** to **Superset** follows using Lemma 30.

Theorem 34. *Game* (A, B, C) *has a NE in* $\mathcal{B}_{0.5}$ *if and only if game* (D, E, F) *has a NE where all the strategies played with non-zero probability by players are from* $T_1 = [1:m]$*,* $T_2 = [1:n]$ *and* $T_3 = [1:p]$ *respectively.*

Next theorem follows using Lemmas 30 and 31, and gives a Turing machine reduction from **InBox** to **Subset**.

Theorem 35. *Game* (A, B, C) *has a NE in ball* $\mathcal{B}_{0.5}$ *if and only if* $\exists i \in [m], \exists j \in [n], k \in [p]$ *such that for* $T_1 = \{i\}$*,* $T_2 = \{j\}$ *and* $T_3 = \{k\}$*, game* (D, E, F) *has a NE where all strategies of* T_1, T_2, T_3 *are played with non-zero probability.*

Theorems 33, 34 and 35 together with ETR-hardness of **InBox** in 3-Nash, and Theorem 25 gives the next result.

Theorem 36. *The problems of* **MaxPayoff***,* **Subset** *and* **Superset** *are ETR-complete in* 3*-player games.*

A 3-player game can be reduced to a k-player game trivially, without changing its set of NE, by adding $k-3$ dummy players with one strategy each (and payoff tensor $A_i = [h]$ to get reduction for **MaxPayoff**). And therefore, the next theorem follows from Theorem 37.

Theorem 37. *Given a* k*-player game* $(A_1, \ldots, A_k)$*, for a constant* $k \geq 3$*, the problems of* **MaxPayoff***,* **Subset** *and* **Superset** *are ETR-complete.*

C.1 MaxPayoff to NonUnique

In this section we reduce **MaxPayoff** to **NonUnique** in a 3-player game. Let (A, B, C) be a given game, and for a given rational number $h > 0$, we are asked to check if it has a NE where all three players get payoff at least h. We will reduce this problem to checking if game (D, E, F) has more than one equilibrium. Tensors A, B, C are of $m \times n \times p$-dimensional, where m, n, p are number of strategies of player 1, 2, 3 respectively. Let $m' = m+1, n' = n+1, p' = p+1$, then D, E, F are of dimension $m' \times n' \times p'$, where

$$\forall i \in [m], j \in [n], k \in [p], \quad D_{ijk} = A_{ijk},\ E_{ijk} = B_{ijk},\ F_{ijk} = C_{ijk}$$
$$\forall j \in [n'], k \in [p'],\ D_{m'jk} = h; \quad \forall i \in [m'], k \in [p'],\ E_{in'k} = h$$
$$\forall i \in [m'], j \in [n'],\ F_{ijp'} = h$$

Rest of the entries in D, E, F are set to zero. Basically, we added one extra strategy for each player and made sure that the player gets payoff h when she plays this extra strategy regardless of what others play. Thus, the next lemma follows by construction.

Lemma 38. *Let $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ be a strategy profile for game (D, E, F), and $\boldsymbol{x} = \boldsymbol{x}'(1:m), \boldsymbol{y} = \boldsymbol{y}'(1:n)$ and $\boldsymbol{z} = \boldsymbol{z}'(1:p)$. Then,*

- $\pi_1^D(m', \boldsymbol{y}', \boldsymbol{z}') = h$, $\pi_2^E(\boldsymbol{x}', n', \boldsymbol{z}') = h$, *and* $\pi_3^F(\boldsymbol{x}', \boldsymbol{y}', p') = h$.
- $\forall i \in [m],\ \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z})$. $\forall j \in [n],\ \pi_2^E(\boldsymbol{x}', j, \boldsymbol{z}') = \pi_2^B(\boldsymbol{x}, j, \boldsymbol{z})$. $\forall k \in [p],\ \pi_3^F(\boldsymbol{x}', \boldsymbol{y}', k) = \pi_3^C(\boldsymbol{x}, \boldsymbol{y}, k)$.

Next we show that game (D, E, F) has a trivial pure NE where all player plays their extra strategy.

Lemma 39. *Pure-strategy profile (m', n', p') is a NE of game (D, E, F)*

Proof. When players two and three are playing strategy n' and p' respectively, then $\forall i \in [m]$ payoff $D_{in'p'}$ of the first player is zero, while $D_{m'n'p'} = h > 0$. Therefore playing m' is the best response for her. Similarly, we can argue for players two and three. □

Except for the trivial NE established in Lemma 39 if game (D, E, F) another equilibrium, then we need to construct a solution of **MaxPayoff** in game (A, B, C).

Lemma 40. *If $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}') \neq (m', n', p')$ is a Nash equilibrium of game (D, E, F), then $(\eta(\boldsymbol{x}), \eta(\boldsymbol{y}), \eta(\boldsymbol{z}))$ is a NE of game (A, B, C) with payoff at least h to each player, where $\boldsymbol{x} = \boldsymbol{x}'(1:m), \boldsymbol{y} = \boldsymbol{y}'(1:n)$ and $\boldsymbol{z} = \boldsymbol{z}'(1:p)$.*

Proof. First we show that $\sigma(\boldsymbol{x}), \sigma(\boldsymbol{y}), \sigma(\boldsymbol{z}) > 0$. To the contrary suppose $\boldsymbol{z} = \boldsymbol{0}$ and wlog $\boldsymbol{x} \neq \boldsymbol{0}$. Then, $z'_{p'} = 1$, and $\exists i \in [m],\ x'_i > 0$ with payoff $\pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z}) = 0$ (Lemma 38), a contradiction because player one will deviate to m' which always fetches payoff $h > 0$. Similar contradiction can be derived if $\sigma(\boldsymbol{y}) = 0$ or $\sigma(\boldsymbol{x}) = 0$.

We will show that $\eta(\boldsymbol{x})$ is a best response of the first player when other two are playing $\eta(\boldsymbol{y})$ and $\eta(\boldsymbol{z})$ respectively in (A, B, C), and that her payoff is at least h. Argument for other players follow similarly. Let $\lambda = \max_{s \in [m]} \sum_{j \in [n], k \in [p]} A_{sjk} y_j z_k$. It suffices to show that $\forall i \in [m],\ x_i > 0 \Rightarrow \sum_{j \in [n], k \in [p]} A_{ijk} y_j z_k = \lambda$ and $\lambda \geq h$, because normalization will increase the payoff of all the pure-strategies, and that too by the same factor.

Let $\lambda' = \max_{i \in [m']} \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}')$, then $\lambda = \lambda'$ because $\exists i \in [m], x_i > 0$ and payoff at i is λ.

$$x_i > 0 \Rightarrow x_i' > 0 \Rightarrow \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}') = \lambda' \Rightarrow \sum_{j \in [n], k \in [p]} A_{sjk} y_j z_k = \lambda$$

Now since each player gets payoff h from their last strategy in game (D, E, F) (Lemma 38), other strategies played with non-zero probabilities have to fetch payoff at least h and hence $\lambda = \lambda' \geq h$ follows. □

We also need to establish that if game (A, B, C) has a feasible solution for **MaxPayoff** then game (D, E, F) has more than one equilibrium.

Lemma 41. *If $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a NE of (A, B, C) where every player gets payoff at least h, then $((\boldsymbol{x}|0), (\boldsymbol{y}|0), (\boldsymbol{z}|0))$ is a NE of game (D, E, F).*

Proof. Let $\boldsymbol{x}' = (\boldsymbol{x}|0), \boldsymbol{y}' = (\boldsymbol{y}|0)$ and $\boldsymbol{z}' = (\boldsymbol{z}|0)$. We will show that $\boldsymbol{x}'$ is a best response for player one against $\boldsymbol{y}', \boldsymbol{z}'$ in (D, E, F), and cases for other two players follow similarly. Let $\lambda = \max_{i \in [m]} \pi_1^A(i, \boldsymbol{y}, \boldsymbol{z})$ and $\lambda' = \max_{i \in [m']} \pi_1^D(i, \boldsymbol{y}', \boldsymbol{z}')$. Since $\lambda \geq h$ and $\pi_1^D(m', \boldsymbol{y}', \boldsymbol{z}') = h$ (Lemma 38) we get $\lambda = \lambda'$, and the lemma follows. □

Using Lemmas 39, 40 and 41, we get the next theorem.

Theorem 42. *Game (A, B, C) has a NE where every player gets payoff h iff game (D, E, F) has more than one equilibrium.*

As argued in Section 3.1 a 3-player game can be trivially reduced to a k-player game by adding $k - 3$ dummy agents. Therefore, next theorem follows using Theorems 25, 36 and 42.

Theorem 43. *Given a k-player game $(A_1, \ldots, A_k)$, for a constant $k \geq 3$, the problems of* **NonUnique** *is ETR-complete.*

D Symmetric k-Nash: Containment in FIXP$_a$

Next we show that symmetric k-Nash, for a constant k, is in FIXP, and consequently strong approximation is in FIXP$_a$. Let the given game be represented by tensor A and let the set of pure strategies of players be S. At a symmetric NE all players play the same mixed-strategy. Consider a function $F : \Delta \to \Delta$ as follows, where $\boldsymbol{x}' = F(\boldsymbol{x})$ for an $\boldsymbol{x} \in \Delta$:

$$\forall s \in S, \quad x_s' = \frac{x_s + \max\{\pi^A(s, \boldsymbol{x}) - \pi^A(\boldsymbol{x}), 0\}}{1 + \sum_s \max\{\pi^A(s, \boldsymbol{x}) - \pi^A(\boldsymbol{x}), 0\}} \tag{7}$$

Nash [15] proved that fixed-points of F are exactly the symmetric NE of game A.

Theorem 44. *The problem of computing symmetric NE in a symmetric k-player game is in FIXP for a constant k is in FIXP, and corresponding strong approximation is in* FIXP_a.

Proof. The operations used in defining F are $+, -, *, /$ and max. Further, domain of F is convex and compact, and function is well-defined over the domain. Thus, finding fixed-points of F is in FIXP by definition. Since, description of F is $O(size(A))$ it together with Nash's result [15] imply that finding symmetric NE of A is also in FIXP. Further, for a given $\epsilon > 0$ if $\boldsymbol{x}$ is ϵ-near to an actual fixed-point $\boldsymbol{x}^*$, i.e., $|\boldsymbol{x} - \boldsymbol{x}^*|_\infty < \epsilon$, then $\boldsymbol{x}$ is also a strong approximate symmetric NE of game A. Containment in FIXP_a follows. □

E Missing proofs of section 4

Proof of Lemma 15. We will first show that G has no symmetric NE of support one or two. This involves a case analysis of which we present one representative case each. First observe that $(\alpha, \beta, \gamma) = (1, 0, 0)$ cannot be a symmetric NE, since player 1 should play $(0, 0, 1)$ if the other two players play the given strategy. Next consider the strategy $(\alpha, \beta, 0)$ with the first two components non-zero. Because of the four zeros in the upper left corner of the second matrix, player 1 will be better off playing the third strategy instead of the second strategy. Hence any symmetric NE of G must be of full support, proving the first claim.

Since the payoffs from three strategies of G are respectively $\beta^2 + 2a\beta\gamma$, $\gamma^2 + 2b\alpha\gamma$, and $\alpha^2 + 2c\alpha\beta$, and all three strategies are played, the second claim follows. □

Proof of Lemma 16. Let $\alpha = \sigma(\boldsymbol{x})$, $\beta = \sigma(\boldsymbol{y})$ and $\gamma = \sigma(\boldsymbol{z})$. Clearly, the payoffs from three strategies of G are respectively $\beta^2 + 2a\beta\gamma = \beta^2 + 2a'$, $\gamma^2 + 2b\alpha\gamma = \gamma^2 + 2b'$, and $\alpha^2 + 2c\alpha\beta = \alpha^2 + 2c'$. Observe that these are also the best payoffs among strategies $[1 : m], [m + 1 : m + n]$ and $[m + n + 1 : l]$ respectively in game D. Let the maximum among these three be λ. Then, we have

$$\alpha = \sigma(\boldsymbol{x}) > 0 \Rightarrow \exists i \leq m,\ x_i > 0 \Rightarrow \pi^D(i, \boldsymbol{d}) = \beta^2 + 2a' = \beta^2 + 2a\beta\gamma = \lambda$$

Similarly, we can show that if $\beta > 0$ then payoff at second strategy is λ, and if $\gamma > 0$ then third gives λ. Hence (α, β, γ) is a symmetric NE of game G. □

Proof of Lemma 17. Let $\boldsymbol{x}' = \eta(\boldsymbol{x}), \boldsymbol{y}' = \eta(\boldsymbol{y})$ and $\boldsymbol{z}' = \eta(\boldsymbol{z})$. These are well-defined because lemmas 15 and 16 imply the following claim.

Claim. In any symmetric NE $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$ of game D, all three components $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ of the strategy profile are non-zero.

We will show that $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ satisfies conditions (3) characterizing NE of game (A, B, C). We do this for the first condition, the rest two follow similarly. Let λ

denote the maximum payoff of an agent in the symmetric game D when others are playing $\boldsymbol{d}$. For strategy $s \in S_1$ of the first player, we have,

$$\begin{aligned} x'_s > 0 \Rightarrow x_s > 0 \\ &\Rightarrow \pi^D(s, \boldsymbol{d}) = \lambda \ \ \text{(Using (5) and (3))} \\ &\Rightarrow \pi^D(s, \boldsymbol{d}) \geq \pi^D(s', \boldsymbol{d}), \ \ \forall s' \leq m \\ &\Rightarrow \textstyle\sum_{j \in [n], k \in [p]} A_{sjk} y'_j z'_k \geq \sum_{j \in [n], k \in [p]} A_{s'jk} y'_j z'_k, \forall s' \leq m \end{aligned}$$

□

Proof of Lemma 18. Clearly, $a = \max_{i \in S_1} \sum_{j,k} A_{ijk} y_j z_k$, $b = \max_{j \in S_2} \sum_{i,k} B_{ijk} x_i z_k$ and $c = \max_{i,j} C_{ijk} x_i y_j$. Let $\boldsymbol{x}' = \alpha \boldsymbol{x}$, $\boldsymbol{y}' = \beta \boldsymbol{y}$ and $\boldsymbol{z}' = \gamma \boldsymbol{z}$. Since, $\alpha, \beta, \gamma > 0$ (Lemma 15), we have $\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}' \neq 0$. In the symmetric game D, let $a' = \max_{s \leq m} \pi^D(s, \boldsymbol{d}) = \beta^2 + 2a\beta\gamma$, $b' = \max_{m < s \leq m+n} \pi^D(s, \boldsymbol{d}) = \gamma^2 + 2b\alpha\gamma$, and $c' = \max_{m+n < s \leq l} \pi^D(s, \boldsymbol{d}) = \alpha^2 + 2c\alpha\beta$. Note that a', b', c' are payoffs from the three strategies at (α, β, γ) in game G. Since (α, β, γ) is a NE of G, we have $a' = b' = c'$ (using Lemma 15).

As $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a NE of game (A, B, C), we get

$$\forall i \in [m], \ x'_i > 0 \Rightarrow x_i > 0 \Rightarrow \sum_{j,k} A_{ijk} y_j z_k = a \Rightarrow \pi^D(i, \boldsymbol{d}) = a'$$

Similarly we get, $\forall j \in [n], \ y'_j > 0 \Rightarrow \pi^D(m+j, \boldsymbol{d}) = b'$, and $\forall k \in [p], \ z'_k > 0 \Rightarrow \pi^D(m+n+k, \boldsymbol{d}) = c'$. Lemma follows using the fact that $a' = b' = c'$. □

Proof of Lemma 20 Theorem 12 establishes that checking if game (A, B, C) has a NE where strategies in $T_i \subset S_i, \ i = 1, 2, 3$ are played with non-zero probability is ETR-complete. Let $l = m + n + p$. Construct a symmetric game D of dimension $l \times l \times l$ from G of (4) by blowing it up and replacing a, b and c with A, B, and C respectively. Formally construct D is as follows:

$$\begin{array}{r} D_{stu} = \\ 1 \leq s,t,u \leq l \end{array} \begin{cases} A_{s(t-m)(u-m-n)} & \text{if } s \leq m \ \& \ m < t \leq m+n \ \& \ m+n < u \leq l \\ A_{s(u-m)(t-m-n)} & \text{if } s \leq m \ \& \ m < u \leq m+n \ \& \ m+n < t \leq l \\ B_{t(s-m)(u-m-n)} & \text{if } t \leq m \ \& \ m < s \leq m+n \ \& \ m+n < u \leq l \\ B_{u(s-m)(t-m-n)} & \text{if } u \leq m \ \& \ m < s \leq m+n \ \& \ m+n < t \leq l \\ C_{t(u-m)(s-m-n)} & \text{if } t \leq m \ \& \ m < u \leq m+n \ \& \ m+n < s \leq l \\ C_{u(t-m)(s-m-n)} & \text{if } u \leq m \ \& \ m < t \leq m+n \ \& \ m+n < s \leq l \\ 1 & \text{if } s \leq m \ \& \ m < t = u \leq m+n, \\ 1 & \text{if } m < s \leq m+n \ \& \ m+n < t = u \leq l \\ 1 & \text{if } m+n < s \leq l \ \& \ t = u \leq m \\ 0 & \textit{Otherwise.} \end{cases} \quad (8)$$

Let $T = T_1 \cup \{j + m \mid j \in T_2\} \cup \{k + m + n \mid k \in T_3\}$. Using Theorem 19 it follows that game (A, B, C) has a NE where strategies of T_i are played with positive probability if and only if game D has a symmetric NE where strategies of T are played with positive probability. Since size of D is $O(size(A,B,C))$, ETR-hardness follows.

Let $l = |S|$, then the problem reduces to checking if there exists a vector $\boldsymbol{d}$ that satisfies the following:

$$\forall t \in T, d_t > 0 \quad \forall s \in S \setminus T, d_s \geq 0,$$
$$\forall s \in S, \sum_{j\in[l],k\in[l]} D_{ijk} d_j d_k \leq \pi, \quad d_s(\sum_{j\in[l],k\in[l]} D_{ijk} d_j d_k - \pi) = 0$$

The last condition ensures that $(\boldsymbol{d},\boldsymbol{d},\boldsymbol{d})$ satisfies (3), and constitutes a symmetric NE of game D. Thus containment in ETR follows. □

F FIXP$_a$-completeness for symmetric 3-Nash

Using the reduction from 3-Nash to symmetric 3-Nash established by Theorem 19, together with FIXP$_a$-completeness for 3-Nash [7], in this section we show that symmetric 3-Nash is also FIXP$_a$-complete. For this we need to compute a strategy profile $(\boldsymbol{x}',\boldsymbol{y}',\boldsymbol{z}')$ that is ϵ-near to an actual equilibrium of (A,B,C), given a symmetric profile $\boldsymbol{d}$ ϵ'-near to a symmetric NE $\boldsymbol{d}^*$ of D, where distances are measured in l_∞ norm.

In reduction of Theorem 19, obtaining solution of (A,B,C) involves e.g., dividing $\boldsymbol{x}$ by $\sigma(\boldsymbol{x})$. If the latter is very small, this may give us a vector that is very far from a solution of (A,B,C), even when $\boldsymbol{d}$ may be close to $\boldsymbol{d}^*$. To get around this next we make sure that $\sigma(\boldsymbol{x})$ is big enough.

Wlog, we assume that all entries of $A,B,C \in [0,0.1]$, as adding constants to A,B,C or scaling them by positive constants does not change its set of NE. In that case, payoffs of a player at its NE is in $[0,0.1]$. The a,b,c of Lemma 16 are also in $[0,0.1]$. Thus, if we can lower bound the NE strategy (α,β,γ) of game G with such a,b,c then we get a lower bound on $\sigma(\boldsymbol{x})$, $\sigma(\boldsymbol{y})$ and $\sigma(\boldsymbol{z})$ as desired.

Lemma 45. *If (α,β,γ) is a NE of game G, where $a,b,c \in [0,0.1]$, then $\frac{1}{4} \leq \alpha,\beta,\gamma \leq \frac{1}{2}$.*

Proof. NE (α,β,γ) of G is fully-mixed, and therefore each of the three strategies fetch the same payoff, i.e., $\beta^2 + 2a\beta\gamma = \gamma^2 + 2b\alpha\gamma = \alpha^2 + 2c\alpha\beta$. We show that none of $\alpha,\beta,\gamma < 1/4$, and the upper bound follows. There are two cases for each, and we show them for α. For β and γ they follow similarly.

Case I: $\alpha < {}^1/_4$, and $\beta,\gamma \geq {}^1/_4$.
As $\beta+\gamma \geq {}^3/_4$, wlog let $\beta \geq {}^3/_8$. Then, we have $\beta^2 + a\beta\alpha \geq {}^9/_{64} + {}^{3a}/_{16}$, and $\alpha^2 + 2c\alpha\beta \leq {}^1/_{16} + {}^c/_2$. The above equality gives ${}^9/_{64} + {}^{3a}/_{16} \leq {}^1/_{16} + {}^c/_2 \Rightarrow {}^5/_{64} \leq {}^c/_2 - {}^{3a}/_{16} \Rightarrow c \geq {}^{10}/_{64} \geq 0.1$, a contradiction.

Case II: $\alpha,\gamma < {}^1/_4$, and $\beta > {}^1/_2$.
$\beta^2 + a\beta\gamma \geq {}^1/_4$ and $\gamma^2 + c\alpha\gamma \leq {}^{1+2c}/_{16}$. Thus, we have $4 \leq 1 + 2c \Rightarrow c \geq {}^3/_2$, a contradiction. □

Next we show that strong approximate solution of D maps to a strong approximate solution of (A, B, C), under the mapping of Theorem 19.

Lemma 46. *Let $\boldsymbol{d}^* = (\boldsymbol{x}^*|\boldsymbol{y}^*|\boldsymbol{z}^*)$ be a symmetric Nash equilibrium of game D, and $\boldsymbol{d} = (\boldsymbol{x}|\boldsymbol{y}|\boldsymbol{z})$ be such that $|\boldsymbol{d} - \boldsymbol{d}^*|_\infty \leq \epsilon$. Then, $\left|\frac{x_i}{\sigma(\boldsymbol{x})} - \frac{x_i^*}{\sigma(\boldsymbol{x}^*)}\right| \leq \epsilon'$, $\forall i$; $\left|\frac{y_j}{\sigma(\boldsymbol{y})} - \frac{y_i^*}{\sigma(\boldsymbol{y}^*)}\right| \leq \epsilon'$, $\forall j$; and $\left|\frac{z_k}{\sigma(\boldsymbol{z})} - \frac{z_k^*}{\sigma(\boldsymbol{z}^*)}\right| \leq \epsilon'$, $\forall k$, where $\epsilon = \frac{\epsilon'}{20l^2}$.*

Proof. Lemmas 16 and 45 give us $\frac{1}{4} \leq \sigma(\boldsymbol{x}^*) \leq \frac{1}{2}$. Using this we obtain bounds on $\sigma(\boldsymbol{x})$.

$$\forall i \leq m,\ |x_i - x_i^*| \leq \epsilon \Rightarrow |\sigma(\boldsymbol{x}) - \sigma(\boldsymbol{x}^*)| \leq m\epsilon \Rightarrow \sigma(\boldsymbol{x}^*) - m\epsilon \leq \sigma(\boldsymbol{x}) \leq \sigma(\boldsymbol{x}^*) + m\epsilon$$

Assuming $\epsilon < \frac{1}{20m}$, we get that $\frac{1}{5} \leq \sigma(\boldsymbol{x}) \leq \frac{2}{3}$. Next consider the quantity we wish to bound.

$$\begin{aligned}\left|\frac{x_i}{\sigma(\boldsymbol{x})} - \frac{x_i^*}{\sigma(\boldsymbol{x}^*)}\right| &\leq \frac{9}{2}|x_i \textstyle\sum_{k\neq i} x_k^* - x_i^* \sum_{k \neq i} x_k| \\ &\leq \frac{9}{2}|x_i \textstyle\sum_{k\neq i}(x_k + m\epsilon) - (x_i - m\epsilon)\sum_{k\neq i} x_k| \\ &\leq \frac{9}{2}m\epsilon((m-1)x_i + \textstyle\sum_{k\neq i} x_k) \\ &\leq \frac{9}{2}m^2\epsilon \leq \epsilon'\end{aligned}$$

Similar argument suffices to show $\forall j$, $\left|\frac{y_j}{\sigma(\boldsymbol{y})} - \frac{y_j^*}{\sigma(\boldsymbol{y}^*)}\right| \leq \epsilon'$, and $\forall k$, $\left|\frac{z_k}{\sigma(\boldsymbol{z})} - \frac{z_k^*}{\sigma(\boldsymbol{z}^*)}\right| \leq \epsilon'$. ∎

From Theorem 19 we know that a symmetric NE $\boldsymbol{d}^* = (\boldsymbol{x}^*|\boldsymbol{y}^*|\boldsymbol{z}^*)$ maps to a NE $(\boldsymbol{x}'^*, \boldsymbol{y}'^*, \boldsymbol{z}'^*) = (\eta(\boldsymbol{x}^*), \eta(\boldsymbol{y}^*), \eta(\boldsymbol{z}^*))$of game (A, B, C). Lemma 46 implies that finding a profile $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ that is ϵ' near to $(\boldsymbol{x}'^*, \boldsymbol{y}'^*, \boldsymbol{z}'^*)$, for any $\epsilon' < 1$ reduces to finding a symmetric profile $\boldsymbol{d}$ that is $\frac{\epsilon}{20l^2}$ near to $\boldsymbol{d}^*$. Clearly, there is such a $\boldsymbol{d}$ with size $poly\{size(A, B, C), \log(\frac{1}{\epsilon'})\}$, and therefore it can be mapped to a solution of (A, B, C) in polynomial time. Since, such an approximation in 3-Nash is FIXP$_a$-hard [7], and symmetric 3-Nash is in FIXP (Theorem 44), the next theorem follows.

Theorem 47. *Symmetric* 3-*Nash is* FIXP$_a$-*complete.*

G Symmetric k-Nash: ETR and FIXP$_a$ Completeness

Building on the construction of Section E, in this section we reduce k-Nash to symmetric k-Nash. Given a k-player game $\mathcal{A} = (A_1, \ldots, A_k)$ we construct a symmetric game D where the set of strategies of each player is $S = \cup_i S_i$, such that NE of game $\mathcal{A}$ maps to symmetric NE of game D, and vice-versa. Note that D will be a k-dimensional tensor with $l = \sum_i m_i$ coordinates in each dimension. First we construct the symmetric game G (Similar to that of (4)), which has now k-players each with k strategies. As players are identical in symmetric games, the payoff of a player from her pure-strategy depends on which strategies are played by how many players; it doesn't matter who played what. Therefore, the non-zero entries of G may be represented as follows, where $a_1, \ldots, a_k$ are non-negative numbers.

$$G(i, i+1, \ldots, i+1) = 1,\ \forall i < k; \qquad G(k, 1, \ldots, 1) = 1$$

$$G(i, \{1, \ldots, i-1, i+1, \ldots, k\}) = a_i, \forall i \leq k$$

Similar to Lemma 15, it follows that all symmetric NE of G are of full support. Next, we can blow up G to construct D, where $(i_1, \ldots, i_k)$th entry is blown up to $m_{i_1} \times \cdots \times m_{i_k}$-dimensional tensor with that entry copied every where except for those corresponding to a_is. In place of a_i we replace A_i after appropriate rotation.

Like Lemma 16 we can show that if $\boldsymbol{d} = (\boldsymbol{x}^1 | \ldots | \boldsymbol{x}^k)$ is a symmetric NE of game D then $(\sigma(\boldsymbol{x}^1), \ldots, \sigma(\boldsymbol{x}^k))$ is a symmetric NE of game G, thereby showing that each of these sums are strictly positive. Here a_i is set to the best payoff achieved among the strategies of $\boldsymbol{x}^i$ divided by $\Pi_{j \neq i} \sigma(\boldsymbol{x}^j)$. Further, $\boldsymbol{d}$ being a NE it ensures that if a coordinate j of $\boldsymbol{x}^i$ is non-zero then payoff from j^{th} strategy, among $\boldsymbol{x}^i$ is the best. This sets the stage to obtain NE of game $\mathcal{A}$ from $\boldsymbol{d}$, namely, $(\eta(\boldsymbol{x}^1), \ldots, \eta(\boldsymbol{x}^k))$ (Similar to Lemma 17).

For the reverse mapping, let $\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k)$ be a NE of game $\mathcal{A}$, and let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be a symmetric NE of G where a_i is set to the payoff agent i receives at the given NE of $\mathcal{A}$. Then, it follows that $\boldsymbol{d} = (\alpha_1 \boldsymbol{x}^1 | \ldots | \alpha_k \boldsymbol{x}^k)$ is a symmetric NE of D. The brief reason is as follows: the best payoff from ith block of strategies is $a'_i = \alpha_{i+1}^{k-1} + (k-1)! a_i \Pi_{j \neq i} \alpha_i$, and $\boldsymbol{x}$ being a NE $\mathcal{A}$ non-zero strategies of $\boldsymbol{x}^i$ fetch best payoff to player i, namely a_i. Hence, in $\boldsymbol{d}$ the strategies played with non-zero probability within block i fetch payoff a'_i. Further, a'_i is also the payoff from ith strategy in game G, and $\boldsymbol{\alpha}$ being a NE with full support, it ensures that all a'_is are same. Thus, in $\boldsymbol{d}$ best payoffs are the same, across blocks, and therefore it is a symmetric NE of game D.

The next theorem follows from the above discussion (of this section).

Theorem 48. *Profile $\boldsymbol{d} = (\boldsymbol{x}^1 | \ldots | \boldsymbol{x}^k)$ is a symmetric NE of game D if and only if $(\eta(\boldsymbol{x}^1), \ldots, \eta(\boldsymbol{x}^k))$ is a NE of game $(A_1, \ldots, A_k)$.*

Using Theorem 48 together with Theorems 26 and 36 we get the following ETR completeness results.

Theorem 49. *For symmetric k-Nash, problems* **Subset** *and* **Superset** *are ETR-complete, where $k \geq 3$ is a constant.*

A normal k-player game can be reduced to $k+1$-player game trivially by adding a dummy player with one strategy and any payoff, and therefore FIXP$_a$-hardness of Theorem 2 extends to k-Nash. However, such a reduction is not possible in case of symmetric games, because the resulting game has to satisfy the symmetry conditions (see Section 2.1). Therefore, FIXP$_a$-hardness for symmetric 3-Nash does not extend to symmetric k-Nash. We show this result using the fact that k-Nash is FIXP$_a$-hard together with Theorem 48.

As done in Section F, we need to lower bound $\sigma(\boldsymbol{x}^i)$ for a given symmetric NE $\boldsymbol{d} = (\boldsymbol{x}^1|\dots|\boldsymbol{x}^k)$. Lower bound of $\frac{1}{4}$ follows by assuming $A_1, \dots, A_k \in [0, \frac{0.1}{k}]$ wlog, as established in Lemma 45. Finally, using this lower bound we can show that if $|\boldsymbol{d} - \boldsymbol{d}^*|_\infty < \epsilon$ where $\boldsymbol{d}^*$ is a symmetric NE, then $|{x^i_s}/\sigma(\boldsymbol{x}^i) - {x^{*i}_s}/\sigma(\boldsymbol{x}^{*i})| < \epsilon'$, $\forall i \in [1:k], \forall s \in S_i$. In other words the strategy profile $(\eta(\boldsymbol{x}^1), \dots, \eta(\boldsymbol{x}^k))$ obtained from $\boldsymbol{d}$ is ϵ'-near to NE $(\eta(\boldsymbol{x^*}^1), \dots, \eta(\boldsymbol{x^*}^k))$ obtained from $\boldsymbol{d}^*$, where $\epsilon = \frac{\epsilon}{20(\max_i |S_i|)^2}$. Thus, FIXP_a-hardness follows for symmetric k-Nash, and we get the next result using Theorem 44.

Theorem 50. *For a constant $k \geq 3$, symmetric k-Nash is* FIXP_a*-complete.*

H Discussion

There is a reduction from symmetric 2-Nash to 2-Nash using the notion of imitation games [13]. Is there an analogous reduction from symmetric k-Nash to k-Nash, for $k \geq 3$? For the case of 2-player games, Papadimitriou [17] asked the complexity of finding a non-symmetric equilibrium in a symmetric game. This was recently shown to be NP-complete [14]. What is the complexity of the analogous question for k-player games, for $k \geq 3$? For the case of 2-player games, the question of counting the number of equilibria, even those satisfying special properties, is typically #P-complete. What is the complexity of analogous questions for k-player games, for $k \geq 3$? Are they PSPACE-complete? Another question is whether our reduction from 3-Nash to symmetric 3-Nash creates a one-to-one correspondence between solutions of the two problems. If so, intractability of counting 3-Nash solutions will carry over to counting symmetric 3-Nash solutions.

Besides the questions studied in this paper, Gilboa and Zemel [10] and Conitzer and Sandholm [5] had studied a number of other questions for the case of 2-player games. These need to be studied for the 3-player case as well. For k-player games, $k \geq 3$, finding an ϵ-Nash equilibrium was shown to be in the class PPAD by [19]. Equilibrium questions that are in this class have admitted complementary pivot algorithms that are practical, e.g., for 2-Nash [12] and for market equilibrium under separable, piecewise-linear concave utility functions [8]. Are there practical algorithms for finding an ϵ-Nash equilibrium in k-player games, $k \geq 3$? Finally, is 3-Nash complete for the class FIXP?