Market Equilibrium under Piecewise Leontief Concave Utilities

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Abstract. Leontief function is one of the most widely used function in economic modeling, for both production and preferences. However it lacks the desirable property of diminishing returns. In this paper, we consider piecewise Leontief concave (p-Leontief) utility function which consists of a set of Leontief-type segments with decreasing returns and upper limits on the utility. Leontief is a special case when there is exactly one segment with no upper limit.

We show that computing an equilibrium in a Fisher market with p-Leontief utilities, even with two segments, is PPAD-hard via a reduction from Arrow-Debreu market with Leontief utilities. However, under a special case when coefficients on segments are uniformly scaled versions of each other, we show that all equilibria can be computed in polynomial time. This also gives a non-trivial class of Arrow-Debreu Leontief markets solvable in polynomial time.

Further, we extend the results of [22,5] for Leontief to p-Leontief utilities. We show that equilibria in case of pairing economy with p-Leontief utilities are rational and we give an algorithm to find one using the Lemke-Howson scheme.

1 Introduction

Market equilibrium is a fundamental concept in mathematical economics and has been studied extensively since the work of Walras [20]. The notion of equilibrium is inherently algorithmic, with many applications in policy analysis and recently in e-commerce [9,12,18]. The Arrow-Debreu (exchange) market model consists of a set of agents and a set of goods, where each agent has an initial endowment of goods and a utility (preference) function over bundle of goods. At equilibrium, each agent buys a utility maximizing bundle from the money obtained by selling its initial endowment and the market clears.

It is customary in economics to assume utility functions to be concave and satisfying the law of diminishing returns. Leontief utility function is a well-studied concave function, where goods are complementary and they are needed in a fixed proportion for deriving a positive utility, for e.g., bread and butter. It is a homogeneous function of degree one, where the utility is multiplied by α

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when the amount of each good is multiplied by α, for any $\alpha > 0$; hence it does not, as such, model diminishing returns. Consider the following example.

Example. Suppose Alice wants to consume sandwiches and for making a sandwich, she needs two slices of bread and one slice of cheese. It seems her utility function for bread and cheese can be modeled as a Leontief function, but it is not appropriate because her utility for the second sandwich is less than the first one due to satiation, and so on.

In this paper, we define piecewise Leontief concave (p-Leontief) utility function, which not only generalizes Leontief but also captures diminishing returns to scale and seems to be more relevant in economics. Further, we derive algorithmic and hardness results for both Fisher* and exchange market models under these functions. A p-Leontief utility function consists of a set of segments, where utility obtained on each segment is as per a Leontief function with a limit on the utility. The extra bundle needed, to obtain another unit of utility on segment k, is strictly more than that on segment k–1†. This puts a natural ordering on the segments, so that function remains concave and captures diminishing returns (see Section 2.4 for the precise definition). Observe that a Leontief function is simply a p-Leontief function with exactly one segment and no upper limit.

Recall the above example. Alice's utility for bread and cheese can be modeled as a p-Leontief function as follows: On the first segment, a unit of utility can be derived by consuming 2 slices of bread and 1 slice of cheese, and the upper limit is 1, *i.e.,* at most 1 unit of utility can be derived on this segment. On the second segment, a unit of utility can be derived by consuming 4 slices of bread and 2 slices of cheese, and the upper limit is 2, and so on.

Since Leontief function is a special case of p-Leontief function, all hardness results for Leontief utilities [5] simply carry over to p-Leontief utilities and we get the following theorem.

Theorem 1 (Hardness of Exchange p-Leontief). *Computing an equilibrium in an exchange market with p-Leontief utilities is PPAD-hard, and all equilibria can be irrational even if all input parameters are rational numbers.*

There is a qualitative difference between the complexity of computing an equilibrium in Fisher and Arrow-Debreu markets under Leontief utilities; while polynomial time in the former case through Eisenberg's convex program [11], it is PPAD-hard in the latter case [5]. In contrast, we show that Fisher is no easier than Arrow-Debreu under p-Leontief utilities and obtain the following theorem.

Theorem 2 (Hardness of Fisher p-Leontief). *Computing an equilibrium in Fisher market with p-Leontief utilities, even with two segments, is PPAD-hard.*

For the above theorem, we essentially give a reduction from exchange p-Leontief with k segments to Fisher p-Leontief with $k + 1$ segments. Further we

*A special case of exchange market model, defined in Section 2.2.

†By *strictly more than*, we mean at least one good is needed in greater amount.

show that when coefficients on segments are uniformly scaled versions of each other, then Fisher market equilibria can be computed in polynomial time. This special case arises in many practical situations, like in the above example of Alice's utility function for bread and cheese. The proportion of bread and cheese on each segment remains 2:1, however her utility per unit of sandwich decreases.

This also gives us a non-trivial class of tractable exchange Leontief markets, where the sum of endowment matrix (W) and Leontief utility coefficient matrix (U) is a constant times all one matrix (see Section 4.1 for details). We note that Fisher is a special case of exchange market model for which W is very special, however an exchange market satisfying our condition does not require W to be special and hence it does not arise from a Fisher market. To the best of our knowledge, apart from the Fisher markets, we are not aware of any other non-trivial tractable classes of exchange Leontief markets.

Pairing economy is a special case of exchange markets, where each agent brings a different good to the market (see Section 2.1 for precise definition). In case of a pairing economy with Leontief utilities, [5] showed that equilibria are rational and they are in one-to-one correspondence with the symmetric Nash equilibria in a symmetric bimatrix game. We extend the results of [5,22] for Leontief to p-Leontief and obtain the following (informal) theorem.

Theorem 3 (Pairing Economy: Rationality and Algorithm). *In a pairing economy with p-Leontief utilities, equilibrium prices are rational if all input parameters are rational. Further computing an equilibrium is PPAD-complete and there is a finite time algorithm to find one using the Lemke-Howson scheme.*

For this, we first characterize equilibrium conditions for exchange market with p-Leontief utilities using the *right* set of variables: a variable to capture price for each good and a variable to capture utility on each segment. In case of pairing economy, these conditions can be divided into two parts. The first part captures the utility on each segment, at equilibrium, as a linear complementarity problem (LCP) formulation, where we use the power of complementarity to ensure that segments are allocated in the correct order. The second part is a linear system of equations of type $A\boldsymbol{p} = \boldsymbol{p}$ in prices given the utilities on each segment. Next we show that the LCP of first part can be solved using the classic Lemke-Howson scheme [14] and then we obtain the equilibrium prices by solving $A\boldsymbol{p} = \boldsymbol{p}$. For this to work, we need a positive solution of $A\boldsymbol{p} = \boldsymbol{p}$, which is guaranteed by the Perron-Frobenius theorem because A turns out to be a positive stochastic matrix.

Related work. Since Leontief is a homogeneous function of degree one, equilibria in a Fisher market with Leontief utilities are captured by Eisenberg's convex program [11] and hence it is computable in polynomial time. Mas-Colell (in [10] by Eaves) gave an example of Leontief economy (both Fisher and exchange), where all equilibria are irrational, even if the input parameters are rational numbers; this discards the possibility of LCP based approach for Leontief economy.

Pairing economy model is used by [5] to show that computing an equilibrium in an exchange market with Leontief utilities is PPAD-hard and it has also been studied in many other settings, for e.g., [22,21,7,23].

There are generalizations of Leontief studied by [22,4]. [22] considered a class of piecewise linear concave (PLC) utility functions and showed that equilibrium in pairing economy is equivalent to solving an LCP. [4] studied market equilibrium under hybrid linear-Leontief utility function. Leontief is a special case in both of them, however these classes are still homogeneous of degree one and do not model the diminishing returns to scale. Hence they are quite different and not comparable with p-Leontief.

2 Preliminaries

2.1 Exchange Market

Exchange is a most fundamental market model and it is extensively studied since the work of Walras [20]. An exchange market consists of a set of agents $\mathcal{A}$ and a set of goods $\mathcal{G}$. Let $m \stackrel{\text{def}}{=} |\mathcal{A}|$ and $n \stackrel{\text{def}}{=} |\mathcal{G}|$. Each agent comes to the market with an initial endowment of goods, where W_{ij} is the amount of good j with agent i, and a utility function $u_i : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$ over bundle of goods. Given prices of goods $\boldsymbol{p} = (p_1, \dots, p_n)$, where p_j is the price of good j, each agent i earns $\sum_{j \in \mathcal{G}} W_{ij} p_j$ by selling its initial endowment and buys a (optimal) bundle which maximizes its utility function from the earned money. At *equilibrium* prices, market clears, i.e., demand of each good matches with its supply.

It is customary in economics to assume utility functions to be non-negative, non-decreasing and concave; non-negative and non-decreasing because of free-disposal property, and concave to model the diminishing returns. The celebrated Arrow-Debreu theorem [1] shows that market equilibrium exists for a very general class of utility functions under some mild conditions. Further, we note that equilibrium prices in this case is scale invariant, i.e., if $\boldsymbol{p}$ is equilibrium prices, then so is $\alpha \boldsymbol{p}, \forall \alpha > 0$. And, it is without loss of generality, to assume that the total initial endowment of each good, i.e., $\sum_i W_{ij} = 1, \forall j \in \mathcal{G}$, is unit[‡].

Pairing Economy. In pairing economy, the initial endowment of each agent is a good which is different than the goods brought by other agents – agents and goods are paired up. In this case, W is an identity matrix, i.e., $W = I$.

2.2 Fisher Market

Fisher market model was defined by Irving Fisher in 1891 [2], where unlike exchange market, buyers and sellers are two different entities. A Fisher market consists of a set of agents $\mathcal{A}$ and a set of goods $\mathcal{G}$. Each agent i has money E_i, and a utility function $u_i : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$ over bundle of goods. Let Q_j denotes

[‡]This is like redefining the unit of goods by appropriately scaling utility parameters.

the quantity of good j in the market. Given prices of goods $\boldsymbol{p} = (p_1, \dots, p_n)$, where p_j is the price of good j, each agent i buys a (optimal) bundle which maximizes its utility function subject to its budget constraints. At *equilibrium* prices, market also clears, i.e., demand of each good matches with its supply.

It is a special case of exchange market: set $W_{ij} = Q_j \frac{E_i}{\sum_i E_i}, \forall (i,j)$, and keep the u_i's unchanged.

2.3 Leontief Utility Function

Leontief is a well studied utility function in economics, where goods are complementary and they are needed in a fixed proportion for deriving positive utility, for e.g. bread and butter. Formally, from a bundle $\boldsymbol{x}_i = (x_{i1}, \dots, x_{in})$ of goods, a Leontief utility function of agent i may be defined as

$$u_i(\boldsymbol{x}_i) = \min_j \{\frac{x_{ij}}{U_{ij}}\}.$$

Here agent i derives one unit of utility when it gets U_{ij} amount of each good j. If the utility function of each agent is Leontief, then it can be represented as a matrix $U = [U_{ij}]_{i \in \mathcal{A}, j \in \mathcal{G}}$, whose i^{th} row $U_i = [U_{ij}]_{j \in \mathcal{G}}$ contains all the coefficients of agent i.

A way to represent Leontief utility function is by choosing x-axis to denote the amount of a good whose coefficient is positive. The amount of the remaining goods are just a fixed proportion of this amount. Fig. 1 depicts a simple example of Leontief utility function on two goods. The goods are required in 2:1 ratio, i.e., one unit of utility is obtained from consuming 2 units of good 1 and 1 unit of good 2.

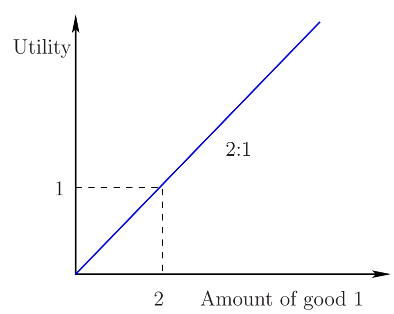

Fig. 1. Leontief utility function

Leontief is a special class of Constant Elasticity of Substitution (CES) [15] utility functions; when CES parameter approaches $-\infty$ we get Leontief. We note that all CES functions are homogeneous of degree one, and so is Leontief, i.e., $u_i(\alpha \boldsymbol{x}_i) = \alpha u_i(\boldsymbol{x}_i), \forall \alpha > 0$. In other words, scaling a bundle by $\alpha > 0$ scales the utility by the same factor. Thus it does not, as such, capture diminishing returns, an important property to model real-life preferences like the example of Alice's utility function in the introduction. To circumvent this, in the next section, we define *piecewise-Leontief concave* utility function, which extends Leontief and allows to model diminishing returns in a logical manner.

2.4 p-Leontief Utility Function

In this section, we define piecewise Leontief concave (p-Leontief) utility function by extending Leontief to capture diminishing returns to scale. Such a function consists of a set of segments (pieces), where the utility derived on each segment is a Leontief function with an upper limit. Further, the coefficient of segments are such that the function remains concave.

Formally, let $u_i : \mathbb{R}_+^n \to \mathbb{R}_+$ be the p-Leontief utility function of agent i with l segments. Since each segment k represents a Leontief function with a limit, it can be represented as $(U_i^k = [U_{ij}^k]_{j \in \mathcal{G}}, L_i^k)$, where U_i^k stores the coefficients of Leontief function and L_i^k stores the limit. The utility u_i from a bundle $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{in})$ is defined as

$$u_i(\boldsymbol{x}_i) = \min_j \left\{ \frac{x_{ij}}{U_{ij}^1}, L_i^1 \right\} + \max\left\{ \min_j \left\{ \frac{x_{ij} - L_i^1 U_{ij}^1}{U_{ij}^2}, L_i^2 \right\}, 0 \right\} + \cdots$$

$$\cdots + \max\left\{ \min_j \left\{ \frac{x_{ij} - \sum_{k=1}^{l-1} L_i^k U_{ij}^k}{U_{ij}^l}, L_i^l \right\}, 0 \right\}.$$

A way to represent p-Leontief utility function is by choosing x-axis to denote the amount of a good whose coefficient is positive. The amount of the remaining goods can be easily obtained from this amount. Fig. 2 depicts a simple example of p-Leontief utility function from two goods. It has three segments; on the first segment, one unit of utility is obtained from 2 units of good 1 and 1 unit of good 2, and the maximum utility that can be derived at this rate is L^1. After that the rate decreases to one unit of utility from 4 units of good 1 and 2 units of good 2 on the second segment, and the maximum utility at this rate is L^2, and so on.

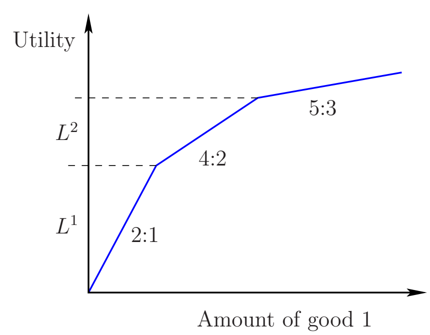

Fig. 2. p-Leontief utility function

Note that $L_i^k U_{ij}^k$ is the amount of good j needed to utilize segment k completely; essentially segments are utilized in order from 1 to l. Further, there is no upper limit on the last segment, i.e., $L_i^l = \infty$. To ensure diminishing returns and concavity, we enforce $U_i^{k+1} \geq U_i^k, \forall k \geq 1$.

Next we show that such a function is indeed concave. For this, consider the following linear program (LP) to compute the utility from a bundle $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{in})$, where β_i^k captures the utility obtained on segment k, and x_{ij}^k

captures the amount of good j allocated on segment k.

$$\begin{aligned}
&\mathbf{max} \sum_{k=1}^{l} \beta_i^k \\
&\sum_{k=1}^{l} x_{ij}^k = x_{ij}, \quad \forall j \in \mathcal{G} \\
&\beta_i^k \leq \frac{x_{ij}^k}{U_{ij}^k}, \ \ 1 \leq k \leq l, \ \ \forall j \in \mathcal{G} \\
&0 \leq \beta_i^k \leq L_i^k, \quad 1 \leq k \leq l.
\end{aligned} \tag{LP}$$

Lemma 4. *For any given $\boldsymbol{x}_i$, optimal solution of (LP) gives $u_i(\boldsymbol{x}_i)$.*

Proof. Since $U_i^{k+1} \geq U_i^k$, $\forall k$, it is easy to check that optimal solution of (LP) will allocate segments in an increasing order, i.e., $\beta_i^{k+1} > 0 \Rightarrow \beta_i^k = L_i^k, \forall k$. Further $\beta_i^k U_{ij}^k = x_{ij}^k$ and it implies that $\beta_i^k = \max\{\min_j\{\frac{x_{ij} - \sum_{a=1}^{k-1} L_i^a U_{ij}^a}{U_{ij}^k}, L_i^k\}, 0\}$. This proves the claim. □

Lemma 5. *p-Leontief is a concave function.*

Proof. We need to show that for any two bundles $\boldsymbol{x}_i$ and $\widetilde{\boldsymbol{x}}_i$, we have $u_i(\lambda \boldsymbol{x}_i + (1-\lambda)\widetilde{\boldsymbol{x}}_i) \geq \lambda u_i(\boldsymbol{x}_i) + (1-\lambda) u_i(\widetilde{\boldsymbol{x}}_i), \forall \lambda \in (0,1)$. Let (β_i^k, x_{ij}^k) and $(\widetilde{\beta}_i^k, \widetilde{x}_{ij}^k)$ be the optimal solutions of (LP) for $\boldsymbol{x}_i$ and $\widetilde{\boldsymbol{x}}_i$ respectively, so $u_i(\boldsymbol{x}_i) = \sum_k \beta_i^k$ and $u_i(\widetilde{\boldsymbol{x}}_i) = \sum_k \widetilde{\beta}_i^k$ using Lemma 4.

Let $\boldsymbol{z}_i = \lambda \boldsymbol{x}_i + (1-\lambda)\widetilde{\boldsymbol{x}}_i$. Consider a candidate $z_{ij}^k = \lambda x_{ij}^k + (1-\lambda)\widetilde{x}_{ij}^k$ and $\gamma_i^k = \min_j\{\frac{z_{ij}^k}{U_{ij}^k}\} = \min_j\{\lambda \frac{x_{ij}^k}{U_{ij}^k} + (1-\lambda)\frac{\widetilde{x}_{ij}^k}{U_{ij}^k}\} \geq \lambda \beta_i^k + (1-\lambda)\widetilde{\beta}_i^k$. Further we have $\gamma_i^k \leq L_i^k$ as both $\beta_i^k, \widetilde{\beta}_i^k \leq L_i^k$. It is easy to check that (γ_i^k, z_{ij}^k) satisfies all the constraints of (LP), and the objective value is $\sum_k \gamma_i^k \geq \lambda \sum_k \beta_i^k + (1-\lambda)\sum_k \widetilde{\beta}_i^k$. Therefore, optimal solution of (LP) at $\boldsymbol{z}_i$ will be at least $\lambda u_i(\boldsymbol{x}_i) + (1-\lambda) u_i(\widetilde{\boldsymbol{x}}_i)$, which proves the claim. □

3 Exchange Market with p-Leontief Utility Functions

In this section, we study exchange markets with p-Leontief utilities. First we show that the problem of computing an equilibrium is hard, and then design a finite time algorithm to compute an equilibrium through an linear complementarity problem (LCP) based approach. Since Leontief is a special case of p-Leontief, the next theorem follows from [5,10].

Theorem 6. – *Checking existence of an equilibrium in p-Leontief exchange markets is NP-hard.*

– *Assuming sufficiency condition*[§]*, computing an equilibrium is PPAD-hard.*
– *All equilibrium prices can be irrational even if all input parameters are rational numbers.*

For pairing economy with Leontief utilities, where each agent brings a different good to the market, [5,22] showed that there exists a rational equilibrium (if one exists), and under a sufficiency condition (see Section 3.1), they are in one-to-one correspondence with symmetric Nash equilibria in a symmetric bimatrix game. Thus equilibrium computation problem in this case is PPAD-complete.

We extend all these results to pairing economy with p-Leontief utilities. We show that there is a rational equilibrium (if one exists) and under the similar sufficiency condition, the problem of finding one is PPAD-complete and it can be obtained using classic Lemke-Howson algorithm [14], which is known to run fast in practice.

Next we begin with the characterization of market equilibrium for pairing economy with p-Leontief utilities.

3.1 Market Equilibrium Characterization

Recall that at market equilibrium, each agent obtains a utility maximizing (optimal) bundle of goods subject to its budget constraints, and market clears. In case of pairing economy, each agent i brings one unit of good i to the market and therefore its budget is the price assigned to its good. Since utility function on each segment is Leontief, goods are consumed in a fixed proportion on a segment, determined by U_{ij}^k's, at an optimal bundle.

Let β_i^k denote the utility obtained on segment k by agent i, and x_{ij}^k denote the amount of good j consumed on segment k by agent i. Note that at least $\beta_i^k U_{ij}^k$ amount of each good j is needed in deriving β_i^k utility. Therefore, at an optimal bundle, we have $x_{ij}^k = U_{ij}^k \beta_i^k,\ \forall(k,i,j)$. Let p_j denote the price of good j. Since x_{ij}^k's can be obtained from β_i^k's, the real set of variables to capture are $\boldsymbol{\beta} = [\beta_i^k]_{i\in\mathcal{A},k\in[l]}$ and $\boldsymbol{p} = [p_j]_{j\in\mathcal{G}}$. Market clearing constraints for each good are:

$$\sum_{i,k} x_{ij}^k = \sum_{i,k} U_{ij}^k \beta_i^k \leq 1, \quad \forall j \in \mathcal{G} \tag{1}$$

Similarly, market clearing constraints for each agent are:

$$\sum_{j,k} x_{ij}^k p_j = \sum_{j,k} U_{ij}^k \beta_i^k p_j = p_i, \quad \forall i \in \mathcal{A} \tag{2}$$

From (1) and (2), we can conclude that if $p_j > 0$ then $\sum_{i,k} U_{ij}^k \beta_i^k = 1$. This can be written as the following complementarity constraint:

$$\sum_{i,k} U_{ij}^k \beta_i^k \leq 1; \quad p_j \geq 0; \quad p_j(\sum_{i,k} U_{ij}^k \beta_i^k - 1) = 0, \quad \forall j \in \mathcal{G}$$

[§]refer to Section 3.1

Further, we need that if $\beta_i^{k+1} > 0$ then $\beta_i^k = L_i^k$, which can be captured as the following complementarity constraint:

$$\beta_i^k \leq L_i^k; \quad \beta_i^{k+1} \geq 0; \quad \beta_i^{k+1}(\beta_i^k - L_i^k) = 0, \quad \forall (k, i)$$

We avoid all-zeros solution by putting the following constraint:

$$\sum_j U_{ij}^l p_j > 0, \quad \forall i \in \mathcal{A}$$

Putting these constraints together, next consider the following conditions in variables $(\boldsymbol{\beta}, \boldsymbol{p})$

$$\forall j \in \mathcal{G}: \quad \sum_{i,k} U_{ij}^k \beta_i^k \leq 1; \quad p_j \geq 0; \quad p_j(\sum_{i,k} U_{ij}^k \beta_i^k - 1) = 0 \tag{3.1}$$

$$\forall i \in \mathcal{A}: \quad \sum_{j,k} U_{ij}^k \beta_i^k p_j = p_i; \quad \beta_i^1 \geq 0 \tag{3.2}$$

$$\forall (k, i): \quad \beta_i^k \leq L_i^k; \quad \beta_i^{k+1} \geq 0; \quad \beta_i^{k+1}(\beta_i^k - L_i^k) = 0 \tag{3.3}$$

$$\forall i \in \mathcal{A}: \quad \sum_j U_{ij}^l p_j > 0 \tag{3.4}$$

Remark 7. Note that the equality in (3.2) is quadratic, which eventually leads to irrational equilibria.

Lemma 8 (Equilibrium Characterization). $(\boldsymbol{\beta}, \boldsymbol{p})$ *gives a market equilibrium of pairing economy iff it satisfies* (3.1)-(3.4).

Proof. It is easy to see that market equilibrium $(\boldsymbol{x}, \boldsymbol{p})$ gives a solution of (3.1)-(3.4), where $\beta_i^k = \frac{x_{ij}^k}{U_{ij}^k}$. Market clearing ensures (3.1) and (3.2) and allocating segments in order ensures (3.3). Since $\sum_j U_{ij}^k p_j \leq \sum_j U_{ij}^{k+1} p_j, \forall k$, if (3.4) is not satisfied for some agent i, then it would have demanded infinite amount of some goods, contradicting market clearing.

For the other direction, let $(\boldsymbol{\beta}, \boldsymbol{p})$ be a solution of (3.1)−(3.4), set $x_{ij}^k = U_{ij}^k \beta_i^k$. Then clearly, (3.1) ensures that no good is sold more than its supply and if a good is under-sold then its price is zero. Further (3.2) ensures that agents do not spend more than their earnings.

Now we only need to show that it gives an optimal bundle to each agent. At an optimal bundle, goods are bought in a fixed proportion on each segment which follows from the fact that $x_{ij}^k = U_{ij}^k \beta_i^k$, and before allocating any amount to segment $k + 1$, segment k should be utilized completely, which is ensured by (3.3).

Finally (3.4) avoids the all-zeros solution, which does not give an equilibrium. □

Remark 9. If we do not put (3.4), then it will capture quasi-equilibrium, where we are not required to assign optimal bundles to zero income agents. Further, (3.1)−(3.4) are general enough to capture equilibrium of exchange market when p_i is replaced with $\sum_j W_{ij} p_j$ in (3.2).

In general, market equilibrium may not exist and checking existence is NP-hard (Theorem 6). Next we show that under a sufficiency condition, equilibrium exists and the problem of computing one is PPAD-complete and we give a finite time algorithm using classic Lemke-Howson scheme for finding one. We also note that it is unlikely to strengthen this, without sufficiency condition, because that will imply NP=co-NP by a result of Megiddo [17].

Sufficiency Conditions Arrow-Debreu [1] gave the following sufficiency conditions for the existence of market equilibrium: (i) $W > 0$, and (ii) each agent is non-satiated, i.e., for each bundle, there is another bundle giving a better utility. Clearly, p-Leontief utility function satisfy non-satiation, however under pairing economy we have $W = I$ and therefore condition (i) is not satisfied. In fact, the following simple example, given in [5], shows that there may not be an equilibrium, in general, for the pairing economy with Leontief utilities.

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Further the sufficiency conditions, based on *economy graph* given in [16], are not applicable for Leontief utility functions. Therefore, [5] assumed $U_{ij} > 0, \forall (i,j)$, and showed existence by reducing the problem to symmetric games. Similarly, we assume

$$U^1_{ij} > 0, \quad \forall (i,j) \tag{4}$$

and under this condition we design an algorithm to compute an equilibrium for pairing economy with p-Leontief utilities, thereby also giving a constructive proof of existence.

3.2 Algorithm

We assume that the conditions of (4) is satisfied for the rest of this section. Note that (4) implies that $U^k_{ij} > 0, \forall (i,j,k)$. Under the sufficiency condition, (3.4) is automatically satisfied at a non-zero equilibrium prices, hence it is not needed. Next we show that β^1_i and p_i are intimately related.

Lemma 10. *At equilibrium $\beta^1_i > 0$ iff $p_i > 0$.*

Proof. Lemma 8 implies that equilibrium satisfies (3.1)−(3.4). If $p_i > 0$ then agent i has money to buy a bundle which can give it positive utility, and hence at equilibrium it will buy something on the first segment and that will imply that $\beta^1_i > 0$. For the other direction, if $p_i = 0$, then agent i has no money to buy anything. Further (3.2) implies $\sum_k \beta_i^k \sum_j U_{ij}^k p_j = p_i$. Since $\boldsymbol{p} \neq 0$ at equilibrium, condition (4) ensures $\sum_j U_{ij}^k p_j > 0, \forall k$, implying that $\beta_i^k = 0, \forall k$, and in particular $\beta_i^1 = 0$. $\square$

Using Lemma 10, and the fact that (3.4) is vacuously satisfied if $\boldsymbol{p} \neq 0$, we partition (3.1)−(3.3) into two parts as follows (like in [5] for Leontief):

$$\text{LCP:} \quad \boxed{\begin{array}{llll} \forall j \in \mathcal{G}: & \sum_{i,k} U_{ij}^k \beta_i^k \leq 1; & \beta_j^1 \geq 0; & \beta_j^1 (\sum_{i,k} U_{ij}^k \beta_i^k - 1) = 0 \\ \forall (k,i): & \beta_i^k \leq L_i^k; & \beta_i^{k+1} \geq 0; & \beta_i^{k+1} (\beta_i^k - L_i^k) = 0 \end{array}} \tag{5}$$

$$\boxed{\forall i \in \mathcal{A}: \quad \sum_{j,k} U_{ij}^k \beta_i^k p_j = p_i; \qquad p_i \geq 0; \qquad \boldsymbol{p} \neq 0} \tag{6}$$

The next lemma follows using Lemma 8 and by the construction.

Lemma 11. *A $(\boldsymbol{\beta}, \boldsymbol{p})$ is an equilibrium of a pairing economy with p-Leontief utility function satisfying (4) iff it satisfies (5) and (6).*

Due to Lemma 11 now our goal is to find a $(\boldsymbol{\beta}, \boldsymbol{p})$ satisfying both (5) and (6). Note that (5) is a linear complementarity problem (LCP) in variables $\boldsymbol{\beta}$. Further, this LCP has a dummy solution $\boldsymbol{\beta} = 0$ which does not correspond to any solution of (6), hence market equilibrium. Using this, consider the algorithm of Table 1 for finding an equilibrium.

Table 1. Algorithm

1. Solve LCP (5) using Lemke-Howson algorithm starting from $\boldsymbol{\beta} = 0$, and obtain a non-zero $\boldsymbol{\beta}$.
2. Set $p_i = 0$ if $\beta_i^1 = 0$
3. Obtain remaining prices using the first equality in (6)

The algorithm in Table 1 works only if the third step gives a positive solution. The next theorem shows that this algorithm indeed gives an equilibrium.

Theorem 12 (Correctness). *Assuming (4), the algorithm in Table 1 computes a market equilibrium of a pairing economy with p-Leontief utility function.*

Proof. First observe that the LCP in (5) is of type $\{A\boldsymbol{y} \leq 1;\ \boldsymbol{y} \geq 0;\ \boldsymbol{y}^T(A\boldsymbol{y} - 1) = 0\}$, which is same as the LCP of finding a symmetric Nash equilibrium in symmetric bimatrix game given by (A, A^T) [5]. Since Lemke-Howson algorithm can find a non-zero $\boldsymbol{y}$ for such an LCP, we conclude that the first step of algorithm in Table 1 gives us a non-zero $\boldsymbol{\beta}$ satisfying (5).

Next we set $p_i = 0$ whenever $\beta_i^1 = 0$ and the remaining set of p_j's is obtained through the equality $\sum_{j,k} U_{ij}^k \beta_i^k p_j = p_i$, and it is a system of linear equalities of type $C\sigma = \sigma$. Under sufficiency condition $U^1_{ij} > 0, \forall(i,j)$, $C > 0$ and stochastic, i.e., sum of every column is 1 due to the complementarity condition of the first condition in (5). By Perron-Frobenius theorem, C has a right positive eigenvector, i.e., $\sigma > 0$, which implies that we get all the remaining prices positive in the third step of the algorithm. The solution $(\boldsymbol{\beta}, \boldsymbol{p})$ obtained from the algorithm clearly satisfy both (5) and (6), hence it is a market equilibrium (Lemma 11). □

A number of results follows as corollaries using Theorem 12 and Lemma 11.

Corollary 13 (Existence). *There exists an equilibrium in a pairing economy with p-Leontief utility function satisfying (4).*

Corollary 14 (Rationality). *Equilibrium prices of pairing economy with p-Leontief utility function are always rational if all input parameters are rational.*

Proof. All the solutions of an LCP are rational when all input parameters are rational [6]. Therefore, solutions of (5) are rational. Further, given a non-zero solution $\boldsymbol{\beta}$ of (5), the third step of the algorithm 1 solves a system of linear equations with rational coefficients and hence computes rational prices that satisfies (6). Thus corollary follows using Lemma 11. □

Theorem 6 shows that the problem of computing an equilibrium is PPAD-hard. Using the rationality and the algorithm next we show it is also in PPAD.

Corollary 15 (PPAD-completeness). *The problem of computing an equilibrium in pairing economy with p-Leontief utilities satisfying (4) is PPAD-complete.*

Proof. Since the problem essentially reduces to finding a symmetric Nash equilibrium in a symmetric game, it is in PPAD. Further, we also have a reverse reduction from game to market, as given in [5] for Leontief utilities, which makes it PPAD-hard. □

In summary, we obtained a finite time algorithm based on the Lemke-Howson scheme to compute an equilibrium in pairing economy with p-Leontief utility function, which runs fast in practice in spite of the problem being PPAD-complete, and thereby extended the results of [5] to significantly general class of utility functions.

4 Fisher Market with p-Leontief Utility Functions

We note that computing an equilibrium in a Fisher market with Leontief utilities is equivalent to solving the Eisenberg's convex optimization problem [11], and hence it is polynomial time computable. In this section we show that this positive result does not extend to markets with p-Leontief utility functions, even if all the utility functions have only *two segments*. Essentially, we show that Fisher is no easier than exchange under p-Leontief, which contrasts with the fact that the behavior of exchange and Fisher under Leontief is entirely different – former is PPAD-hard and latter is polynomial time.

Next we give a reduction from exchange market with Leontief utilities, where $U > 0$ (see Section 2.3 for Leontief utilities), to Fisher market with p-Leontief utility function which has two segments.

Theorem 16 (Reduction from Exchange to Fisher). *An exchange market $\mathcal{M}$ with Leontief utility functions, where $U > 0$, can be reduced to a Fisher market $\mathcal{M}'$ with p-Leontief utility function such that equilibria of $\mathcal{M}$ are in one-to-one correspondence with equilibria of $\mathcal{M}'$ (up to scaling).*

Proof. Let $\mathcal{M}$ is defined by (W, U) (see Section 2 for details), where W is initial endowment matrix and U is Leontief utility function matrix, such that $U_{ij} > 0, \forall(i,j)$. Without loss of generality, we assume that the total quantity of each good is unit, i.e., $\sum_i W_{ij} = 1, \forall j$. We will construct a Fisher market $\mathcal{M}'$ defined by $(\boldsymbol{E}, \boldsymbol{Q}, \widetilde{U}^1, \widetilde{U}^2, L^1)$, where $\boldsymbol{E}$ is initial money vector, $\boldsymbol{Q}$ is the quantity vector, $\widetilde{U}^k$ is Leontief utility function matrix for segment k for $k = 1, 2$, and $L^1 = [L^1_i]_{i \in \mathcal{A}}$ is the bound on maximum utility that can be obtained on the first segment for each agent i. Note that there is no bound on the maximum utility on the last segment.

The idea is to define the first segment, i.e., $\widetilde{U}^1$, such that at any prices $\boldsymbol{p}$, each agent i buys it completely and the remaining money after that is equal to $\sum_j W_{ij} p_j$. Further, we also make sure that exactly unit amount of each good remains after consuming the first segment of each agent entirely. In that case, it is clear that if we define $\widetilde{U}^2 = U$, then on the second segment, Fisher becomes exactly same as exchange, and any equilibrium of $\mathcal{M}'$ will give an equilibrium of $\mathcal{M}$ and vice versa.

Let $m \overset{\text{def}}{=} |\mathcal{A}|$, $U_{max} \overset{\text{def}}{=} \max_{i,j} U_{ij}$, and $\Delta \overset{\text{def}}{=} m U_{max}$. We set

$$\begin{aligned} E_i &= 1, \qquad \forall i \in \mathcal{A} \\ Q_j &= m+1, \qquad \forall j \in \mathcal{G} \\ \widetilde{U}^1_{ij} &= \frac{1}{\Delta}\left(\frac{m+1}{m} - W_{ij}\right), \forall (i,j) \in (\mathcal{A}, \mathcal{G}) \\ L^1_i &= \Delta, \qquad \forall i \in \mathcal{A} \\ \widetilde{U}^2 &= U. \end{aligned}$$

Note that $\widetilde{U}^1_{ij} > 0, \forall(i,j)$ as $W_{ij} \le 1, \forall(i,j)$, and $\widetilde{U}^1_{ij} \le \widetilde{U}^2_{ij}, \forall(i,j)$ due to the choice of Δ. Market clearing condition implies that $(m+1)\sum_j p_j = \sum_i E_i = m$, we have $\sum_j p_j = \frac{m}{m+1}$. The money spent on the first segment by agent i is $\Delta \sum_j \frac{1}{\Delta}(\frac{m+1}{m} - W_{ij}) p_j = 1 - \sum_j W_{ij} p_j$. The remaining money left with agent i is $\sum_j W_{ij} p_j$ to spend on the second segment, which is exactly equal to the money with agent i in $\mathcal{M}$. Further, the total amount of good j needed to completely buy first segment of each agent is $\sum_i (\frac{m+1}{m} - W_{ij}) = m$, hence the remaining amount is $1, \forall j$. Since the utility on the second segment $\widetilde{U}^2$ is same as U, hence each market equilibrium of $\mathcal{M}'$ will give a market equilibrium of $\mathcal{M}$ (up to scaling) and vice versa. □

Remark 17. (a) The above reduction in fact does not require $U > 0$, but without that, $\widetilde{U}$ will not be concave and hence also not p-Leontief.

(b) The reduction can be easily extended to show that exchange market with p-Leontief utility function, which has k segments reduces to Fisher market with p-Leontief, which has $k+1$ segments.

(c) The idea of designing first segment so that it consumes extra money was also used in [19] for proving that Fisher market with separable piecewise-linear and concave utilities is PPAD-hard. However our reduction is more general and does not require the instance of exchange market to be special satisfying *price regulation* property as in [19].

Corollary 18 (Hardness of Fisher p-Leontief). *Computing an equilibrium in a Fisher market with p-Leontief utilities, even with two segments, is PPAD-hard.*

Proof. We note that finding a symmetric Nash equilibrium in a two player symmetric game (A, A^T) is PPAD-hard [3,8]. Without loss of generality, we can assume that $A > 0$ since adding the same constant to all entries does not change the set of Nash equilibria. Further, [5] gave a reduction from the problem of finding a symmetric Nash equilibrium in a symmetric game (A, A^T) to finding a market equilibrium in exchange market with Leontief utilities $(W = I, U = A)$, where I is an identity matrix. Now the claim follows from Theorem 16. □

4.1 A Special Case

In this section, we consider a special case of p-Leontief utility functions, where the coefficients on segments are uniformly scaled versions of each other. Recall that a Leontief utility function of an agent from a bundle $\boldsymbol{x}$ of goods is given by $\min_{j\in\mathcal{G}} \frac{x_j}{U_j}$, where a unit of utility is obtained by consuming U_j amount of each good j. A p-Leontief utility function has a set of Leontief-type segments with upper limits on the utility. Segment k can be described by coefficients $U^k = [U^k_j]_{j\in\mathcal{G}}$. In general, U^k_j's can be arbitrary non-negative numbers satisfying $U^k \leq U^{k+1}$. However when they satisfy

$$\frac{U^{k_1}_{j_1}}{U^{k_1}_{j_2}} = \frac{U^{k_2}_{j_1}}{U^{k_2}_{j_2}}, \quad \forall(k_1, k_2) \text{ and } \forall(j_1, j_2),$$

then we say that coefficients on segments are uniformly scaled versions of each other, however the amount of each good j needed on a segment $k+1$ for a unit of utility is strictly greater than the amount needed on segment k.

This is an interesting special case and seems applicable in many practical situations, like in the example of Alice's utility function for bread and cheese in the introduction.

Lemma 19. *For the special case of p-Leontief utilities, where coefficients on segments are uniformly scaled versions of each other, Fisher market equilibrium can be computed in polynomial time.*

Proof. We note that the Fisher market equilibrium with Leontief utilities can be computed in polynomial time using the Eisenberg's convex program. Now observe that Fisher market equilibrium for Leontief utilities whose coefficients are as per the first segments of p-Leontief utilities, also gives an equilibrium for the special case as well. Here equilibrium prices and allocation map to the exact same values, however the value of utility obtained at equilibrium can be different. □

Non-trivial tractable class of exchange Leontief markets. An exchange market with Leontief utilities is described by an endowment matrix W and a Leontief utility coefficient matrix U. Recall the reduction from Section 4. Using this reduction and Lemma 19, we can solve exchange market with Leontief Utilities in polynomial time, when

$$\frac{1}{\Delta}(\frac{m+1}{m} - W_{ij}) = cU_{ij}, \ \forall(i,j),$$

for a constant $c < 1$[¶]. By redefining the units of goods, we can simplify this condition to

$$W + U = cJ, \tag{7}$$

where c is a constant and J is all one matrix. This gives us a non-trivial class of tractable exchange markets. We note that Fisher is a special case of exchange market model for which W is very special, however an exchange market satisfying (7) does not require W to be special and hence it does not arise from a Fisher market. To the best of our knowledge, apart from the Fisher markets, we are not aware of any other non-trivial tractable classes of exchange Leontief markets.

5 Discussion

We obtain both positive and negative results for p-Leontief utility function; it is as easy as Leontief in case of pairing economy, however Fisher becomes PPAD-hard. An interesting open question is whether exchange market with p-Leontief is harder than Leontief or not when $W \neq I$. Apart from this, we identify an interesting special case of p-Leontief utilities, which is tractable for Fisher markets, which further gives us a non-trivial class of tractable exchange Leontief markets.

Since Leontief function is also used in modeling production capabilities of firms, for e.g. linear activity model [13], it would be interesting and useful to apply p-Leontief function in those settings as well. In this paper, we define p-Leontief as a concave function for modeling preferences of an agent, however concavity can be easily discarded to define a more general function, which can model all sorts of behavior in the production function of a firm, like initially it is increasing returns to scale and then decreasing returns to scale etc. We expect

[¶]This in fact works for any $c > 0$

more applications of piecewise Leontief in future.

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