Markets with Production: A Polynomial Time Algorithm and a Reduction to Pure Exchange

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The classic Arrow-Debreu market model captures both production and consumption, two equally important blocks of an economy, however most of the work in theoretical computer science has so far concentrated on markets without production, i.e., the *exchange economy*. In this paper we show two new results on markets with production.

Our first result gives a polynomial time algorithm for Arrow-Debreu markets under piecewise linear concave (PLC) utilities and polyhedral production sets provided the number of goods is constant. This is the first polynomial time result for the most general case of Arrow-Debreu markets.

Our second result gives a novel reduction from an Arrow-Debreu market $\mathcal{M}$ (with production firms) to an equivalent exchange market $\tilde{\mathcal{M}}$ such that the equilibria of $\mathcal{M}$ are in one-to-one correspondence with the equilibria of $\tilde{\mathcal{M}}$. Unlike the previous reduction by Rader [Rader 1964] where $\tilde{\mathcal{M}}$ is artificially constructed, our reduction gives an explicit market $\tilde{\mathcal{M}}$ and we also get: (i) when $\mathcal{M}$ has concave utilities and convex production sets (standard assumption in Arrow-Debreu markets [Arrow and Debreu 1954]), then $\tilde{\mathcal{M}}$ has concave utilities, (ii) when $\mathcal{M}$ has PLC utilities and polyhedral production sets, then $\tilde{\mathcal{M}}$ has PLC utilities, and (iii) when $\mathcal{M}$ has nested CES-Leontief utilities and nested CES-Leontief production, then $\tilde{\mathcal{M}}$ has nested CES-Leontief utilities.

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1. INTRODUCTION

The classic Arrow-Debreu market model [Arrow and Debreu 1954] is one of the most fundamental and extensively studied model within mathematical economics. It captures both production and consumption, two equally important blocks of an economy and it consists of a set of agents, a set of goods, and a set of production firms. Each agent has an initial endowment of goods and a utility (preference) function over bundle of goods, and each firm has a set of production capabilities to produce a set of goods using a set of raw goods. Agents have profit shares in firms. Given prices of goods, each firm operates at a profit maximizing plan, and each agent buys a most preferred bundle that is affordable from the earned money. At equilibrium, market clears, *i.e.,* demand meets supply. An economy without production firms is called an *exchange* market.

In economics, it is customary to assume that utility functions are concave, and production sets are convex. Since we are in a finite precision model of computation, we will assume that utility functions are piecewise-linear and concave (PLC) and production sets are polyhedral. Clearly by making the pieces fine enough, the approximation to the original utilities and production sets can be made as good as needed. In a celebrated result, Arrow and Debreu [Arrow and Debreu 1954] showed the existence of

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equilibrium for a very general class of markets using Kakutani fixed point theorem (highly non-constructive). This was followed by attempts at finding algorithms that compute market equilibria; especially worth mentioning are algorithms of Scarf [Scarf 1967] and Smale [Smale 1976] to find approximate fixed points. Although these algorithms are known to perform well in practice, and have found important applications, they may take exponential time in the worst case.

A systematic study within theoretical computer science in last thirteen years has led to many remarkable algorithmic and complexity theoretic results for computing equilibria. In the beginning, polynomial time algorithms were obtained for exchange markets with linear utilities [Devanur et al. 2008; Jain 2007; Ye 2008; Orlin 2010; Vegh 2012; Duan and Mehlhorn 2013][1]and for certain other cases [Codenotti et al. 2005; Jain and Varadarajan 2006; Devanur and Kannan 2008]. Then for general utility functions Separable PLC (SPLC) and PLC, the complexity of computing an equilibrium was shown to be PPAD-complete [Chen et al. 2009; Vazirani and Yannakakis 2011; Chen and Teng 2009], and FIXP-complete [Garg et al. 2014a; Garg et al. 2014b] respectively. It is unlikely that a polynomial time algorithm exists for these markets.

For markets with production, [Jain and Varadarajan 2006] gave a polynomial time algorithm for production and utility functions coming from a subclass of nested CES (constant elasticity of substitution) functions. We note that these production functions are constant returns to scale, and they are relatively easy to deal with since there is no positive profit to any firm at an equilibrium. For more general functions, equilibrium computation in markets with SPLC utilities and SPLC production is PPAD-complete [Chen et al. 2009; Garg and Vazirani 2014] and for PLC utilities and polyhedral production sets, it is FIXP-complete [Garg et al. 2014a; Garg et al. 2014b]. We note that SPLC production set is a special subcase of polyhedral production set. [Garg and Vazirani 2014] also gave a polynomial time algorithm for markets with SPLC utilities and SPLC production, when either the number of goods or the number of agents and firms is constant.

In this paper, we show two new results. Our first result gives a polynomial time algorithm for computing an equilibrium in Arrow-Debreu markets with PLC utilities and polyhedral production sets provided the number of goods is constant. This is the first polynomial time result for the most general case of Arrow-Debreu markets. We build on the construction of [Devanur and Kannan 2008], which uses the cell decomposition and LP duality techniques in a remarkable way, and gives a polynomial time algorithm for exchange markets (without production) with PLC utilities provided the number of goods is constant.

Our second result gives a novel reduction from an Arrow-Debreu market $\mathcal{M}$ (with production firms) to an equivalent exchange market $\tilde{\mathcal{M}}$ such that the equilibria of $\mathcal{M}$ are in one-to-one correspondence with the equilibria of $\tilde{\mathcal{M}}$. Earlier, Rader [Rader 1964] gave such a reduction (called *Principle of Equivalence*) for a slightly different market setting (see Section 4.2 for details), where utility functions of agents in the reduced market $\tilde{\mathcal{M}}$ are optimization problems over production constraints of firms. These are "artificially constructed", i.e., they are not in the standard form since variables can take negative values and therefore do not correspond to consumption of goods. In contrast, our reduction is quite different and we construct explicit utility function for each agent in $\tilde{\mathcal{M}}$ which are defined over amount of the goods consumed in $\tilde{\mathcal{M}}$. Further, if

— $\mathcal{M}$ has concave utilities and convex production sets (standard assumption in Arrow-Debreu markets [Arrow and Debreu 1954]), then $\tilde{\mathcal{M}}$ has concave utilities.

[1]Some of these are for Fisher markets [Brainard and Scarf 2000], a special case of exchange economy.

— $\mathcal{M}$ has PLC utilities and polyhedral production sets, then $\tilde{\mathcal{M}}$ has PLC utilities.
— $\mathcal{M}$ has nested CES-Leontief utilities and nested CES-Leontief production, then $\tilde{\mathcal{M}}$ has nested CES-Leontief utilities. We note that these are one of the most widely used functions to model both production and consumption in applied general equilibrium [Shoven and Whalley 1992; de La Grandville 2009]. Also the popular modeling language MPSGE [Rutherford 1999] uses these functions to model production and consumption for equilibrium analysis.

Our reduction is simple and efficient, where all parameters of $\tilde{\mathcal{M}}$ can be obtained in linear time from $\mathcal{M}$. Further, it does not subsume our first result, because we create one new good in $\tilde{\mathcal{M}}$ for each firm in $\mathcal{M}$, hence it gives a polynomial time algorithm using [Devanur and Kannan 2008] only when both the number of goods and the number of firms are constant.

In general Arrow-Debreu markets are considered harder to analyze than exchange markets, due to the complexities introduced by firms. This is evident from the disparity in the amount of work done on exchange markets and on Arrow-Debreu markets. Our reduction is contrary to these beliefs and can be utilized in understanding Arrow-Debreu markets better using the known results for exchange markets. Various structural results have been extended to the general economy [Sonnenschein 1973; Mas-Colell 1991] from pure exchange using Rader's *implicit* reduction. Our *explicit* reduction, in addition, facilitates equivalent description of utility function for a given production set. Using this we can extend computational as well as structural results for exchange market with specific utility functions to general markets with specific production functions.

Technical Details. For the first result, we need to capture: (i) optimal production schedules of each firm, (ii) optimal bundles of each agent, and (iii) market clearing conditions. In comparison with exchange markets, we not only have one additional task of handling optimal production schedule of each firm, but also now agents optimal bundles depend on the profit earned by firms because they own a share of that profit. Hence the construction of [Devanur and Kannan 2008], which captures agents optimal bundles, needs to be appropriately extended. And clearly market clearing conditions need to handle the fact that amount of goods available is not a constant anymore and depends on the production.

Essentially like in [Devanur and Kannan 2008], we will partition the price space with a set of hyperplanes/polynomial surfaces, and then in each cell of the partition, we will check if there is a price vector which gives an equilibrium. Since the number of goods is constant, price space is constant dimensional and we put polynomial many hyperplanes/polynomial surfaces, which partition the space into polynomially many non-empty cells and for each cell we construct a polynomial time query. Hence we get a polynomial time algorithm.

As discussed above, we cannot directly capture optimal bundles of agents because the budget constraints depend on the profit earned by firms and profit depends on the production schedules used by firms. If we introduce variables for the produced and used goods of firms optimization programs, then there will be too many variables (as number of firms need not be a constant) and above all, these variables will be common to both agents and firms optimization programs. To overcome these difficulties, first we partition the price space with hyperplanes obtained from firms optimization programs, where each cell in the partition captures the set of optimal production schedules of each firm and in turn their profit as a linear function in price variables.

Using the profit earned by each firm in a particular cell, we next capture the agents optimal bundles by further partitioning this cell into subcells, using a similar construction as in [Devanur and Kannan 2008], where each subcell has information about the optimal bundles of each agent. Next we need to check if this subcell also satisfy market clearing conditions. Note that unlike in exchange, the amount of goods available to agents is not a constant anymore due to production. We use a novel application of LP duality to ensure that production is accounted for in market clearing. Using this, we further partition the price space, and finally we obtain a polynomial time query to be checked in each cell.

The reduction of the second result maps agents and goods of $\mathcal{M}$ into agents and goods of $\tilde{\mathcal{M}}$. For each firm in $\mathcal{M}$, it creates a new agent and a new good in $\tilde{\mathcal{M}}$. The initial endowment and utility function of each new agent is set in such a way so that it functions exactly like the corresponding firm in $\mathcal{M}$. One possibility is that agents consume the raw goods, and bring the produced good as part of the initial endowment to sell in the market. One problem is how to produce variable amount of a good from a fixed endowment. Another major difficulty is to convert the production set, which is a *correspondence*, into a utility *function*. And the third is to transfer profit of a firm from its agent to another agent, which we do by adding an extra good whose price captures the profit. We note that equilibria of $\mathcal{M}$ is just a projection of the equilibria of $\tilde{\mathcal{M}}$.

2. THE ARROW-DEBREU MARKET MODEL

The Arrow-Debreu market [Arrow and Debreu 1954] consists of a set $\mathcal{G}$ of divisible goods, a set $\mathcal{A}$ of agents and a set $\mathcal{F}$ of firms. Let $m \stackrel{\text{def}}{=} |\mathcal{A}|, n \stackrel{\text{def}}{=} |\mathcal{G}|$ and $l \stackrel{\text{def}}{=} |\mathcal{F}|$.

The production capabilities of a firm is defined by a set of production schedules. If a firm can produce a bundle $\boldsymbol{x}^p$ of goods using bundle $\boldsymbol{x}^r$ as raw material, then such a production schedule defines a production possibility vector (PPV) $(\boldsymbol{x}^p - \boldsymbol{x}^r)$. The set of PPVs of a firm determines its production capabilities. Let $\mathcal{S}^f \in \mathcal{R}^n$ denote the PPV set of firm f. Following are the standard and natural assumptions on $\mathcal{S}^f$ [Arrow and Debreu 1954].

(1) Set $\mathcal{S}^f$ is closed and convex, and contains the origin.
(2) The set of produced goods and raw goods of a firm are disjoint. Define $\mathcal{R}^f \stackrel{\text{def}}{=} \{j \in \mathcal{G} \mid v_j < 0,\ \boldsymbol{v} \in \mathcal{S}^f\}$ to be the set of raw goods and $\mathcal{P}^f \stackrel{\text{def}}{=} \{j \in \mathcal{G} \mid v_j > 0,\ \boldsymbol{v} \in \mathcal{S}^f\}$ to be the set of produced goods, then $\mathcal{R}^f \cap \mathcal{P}^f = \emptyset$.
(3) *Downward close* - Adding to raw material does not decrease the production, i.e., if $\boldsymbol{v} \in \mathcal{S}^f$, and $\boldsymbol{w} \leq \boldsymbol{v}$, while $w_j \geq 0, \forall j \in \mathcal{P}^f$ then $\boldsymbol{w} \in \mathcal{S}^f$.
(4) *No production out of nothing* - $\{\oplus_{f \in \mathcal{F}} \mathcal{S}^f\} \cap \mathbb{R}^n_+ = \mathbf{0}$.

The goal of a firm is to produce as per a profit maximizing (optimal) schedule. Firms are owned by agents: Θ^i_f is the profit share of agent i in firm f such that $\forall f \in \mathcal{F},\ \sum_{i \in \mathcal{A}} \Theta^i_f = 1$.

Each agent i comes with an initial endowment of goods; W^i_j is amount of good j with agent i. The preference of agent i is given by a utility function $u_i : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$. Each agent wants to buy a (optimal) bundle of goods that maximizes her utility to the extent allowed by her earned money – from initial endowment and profit shares in the firms.

Let $\boldsymbol{p} \in \mathbb{R}^n$ denote the prices of goods, where p_j is price of good j. Given $\boldsymbol{p}$, let $OPT^i(\boldsymbol{p})$ and $OPT^f(\boldsymbol{p})$ respectively denote the set of optimal bundles of agent i and the set of optimal production schedules of firm f. Let $\boldsymbol{x}^i \in \mathbb{R}^n$ denote the assignment of goods to agent i. Let $(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p}) \in (\mathcal{R}^f, \mathcal{P}^f)$ denote the assignment of raw and produced goods to firm f such that $(\boldsymbol{x}^{f,p} - \boldsymbol{x}^{f,r}) \in \mathcal{S}^f$.

If there is an assignment $\boldsymbol{x}^i \in OPT^i(\boldsymbol{p})$ for each agent i, and $(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p}) \in OPT^f(\boldsymbol{p})$ for each firm f so that there is neither deficiency nor surplus of any good, then such prices are called *market clearing* or *market equilibrium* prices. Let $\boldsymbol{x} = \{\boldsymbol{x}^i, \boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p} : i \in \mathcal{A}, f \in \mathcal{F}\}$ denote an assignment of the market. Formally,

Definition 2.1 (*Market Equilibrium*). Given an Arrow-Debreu market $\mathcal{M}$, $(\boldsymbol{x}, \boldsymbol{p})$ is an equilibrium of $\mathcal{M}$ if $\boldsymbol{x}^i \in OPT^i(\boldsymbol{p}), \forall i \in \mathcal{A}$, $(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p}) \in OPT^f(\boldsymbol{p}), \forall f \in \mathcal{F}$ and market clears, i.e., $\sum_{i \in \mathcal{A}} x^i_j + \sum_{f \in \mathcal{F}} x^{f,r}_j \leq \sum_{i \in \mathcal{A}} W^i_j + \sum_{f \in \mathcal{F}} x^{f,p}_j, \forall j \in \mathcal{G}$.

The market equilibrium problem is to find such prices when they exist. In a celebrated result, Arrow and Debreu [Arrow and Debreu 1954] proved that market equilibrium always exists under some mild conditions, however the proof is non-constructive and uses heavy machinery of Kakutani fixed point theorem. A well studied restriction of Arrow-Debreu model is *exchange economy*, i.e., markets without production firms.

In economics, it is customary to assume that utility functions are concave, and production sets are convex. Since we are in a finite precision model of computation, we will assume that utility functions are piecewise-linear concave (PLC) and production sets are polyhedral. Clearly by making the pieces fine enough, the approximation to the original utilities and production sets can be made as good as needed.

Polyhedral production sets. Each firm has a production technology to produce a set of goods from a set of *different* raw goods. The polyhedral production set of firm f can be described as

$$\sum_{j \in \mathcal{P}^f} D^f_{jk} x^{f,p}_j \leq \sum_{j \in \mathcal{R}^f} C^f_{jk} x^{f,r}_j + T^f_k, \ \forall k,$$

where D^f_{jk}'s, C^f_{jk}'s and T^f_k's are given non-negative rational numbers, and $x^{f,p}_j$ and $x^{f,r}_j$ denote the amount of good j produced and used respectively. The variables $x^{f,p}_j$ and $x^{f,r}_j$ are respectively defined only for those goods j which can be produced and used by firm f.

Given prices $\boldsymbol{p}$, firm f's profit maximizing plan is a solution of the following linear program (LP):

$$\begin{aligned} \max & \sum_{j \in \mathcal{P}^f} p_j x^{f,p}_j + \sum_{j \in \mathcal{R}^f} p_j x^{f,r}_j \\ & \sum_{j \in \mathcal{P}^f} D^f_{jk} x^{f,p}_j \leq - \sum_{j \in \mathcal{R}^f} C^f_{jk} x^{f,r}_j + T^f_k, \ \forall k \\ & x^{f,p}_j \geq 0, x^{f,r}_j \leq 0 \end{aligned} \tag{1}$$

Remark 2.2. Note that we use non-positive (instead of non-negative) variables to capture amount of raw goods in the above program, which is different from the earlier work [Garg et al. 2014a]. This is crucial in getting the polynomial time algorithm later because now the inner product of every feasible point with the price vector captures the profit at that point.

PLC utilities. The PLC utility function $u_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ of agent i can be described as

$$u_i(\boldsymbol{x}^i) = \min_k \{\sum_j U^i_{jk} x^i_j + T^i_k\},$$

where U^i_{jk}'s and T^i_k's are given non-negative rational numbers. Given prices $\boldsymbol{p}$, agent i's optimal bundle is a solution of the following linear program (LP), where ϕ^f captures the profit of firm f:

$$\begin{aligned} &\max u_i \\ u_i &\le \sum_j U^i_{jk} x^i_j + T^i_k, \ \forall k \\ \sum_j x^i_j p_j &\le \sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f \\ x^i_j &\ge 0, \ \forall j \in \mathcal{G} \end{aligned} \tag{2}$$

3. A POLYNOMIAL TIME ALGORITHM

In this section, we describe a polynomial time algorithm for computing a market equilibrium in Arrow-Debreu markets with piecewise linear concave (PLC) utilities and polyhedral production sets provided the number of goods is constant. We build on the construction of [Devanur and Kannan 2008], which gives a polynomial time algorithm for exchange markets (without production) with PLC utilities provided the number of goods is constant. We need to capture: (i) optimal production schedules of each firm, (ii) optimal bundles of each agent, and (iii) market clearing conditions. In comparison with exchange markets, we not only have one additional task of handling optimal production schedule of each firm, but also now agents optimal bundles depend on the profit earned by firms because they own a share of that profit. Hence the construction of [Devanur and Kannan 2008], which captures agents optimal bundles, needs to be appropriately extended. And clearly market clearing conditions need to handle the fact that amount of goods available is not a constant anymore and depends on the production.

Essentially like in [Devanur and Kannan 2008], we will partition the price space with a set of hyperplanes/polynomial surfaces, and then in each cell of the partition, we will check if there is a price vector which gives an equilibrium. Since the number of goods is constant, price space is constant dimensional and we put polynomial many hyperplanes/polynomial surfaces, which partition the space into polynomially many non-empty cells and for each cell we construct a polynomial time query. Hence we get a polynomial time algorithm.

As discussed above, we cannot directly capture the agents optimal bundles because the budget constraints depend on the profit earned by firms and profit depends on the production schedules used by firms. If we introduce variables for the produced and used goods in the agent's optimization program, then there will be too many variables (as number of firms need not be a constant) and above all, these variables will be common to both agents and firms optimization programs. To overcome these difficulties, first we partition the price space with hyperplanes obtained from firms optimization programs, where each cell in the partition captures the set of optimal production schedules of each firm and in turn their profit as a linear function in price variables. Using the profit earned by each firm in a particular cell, we next capture the agents optimal bundles by further partitioning this cell into subcells, using a similar construction as in [Devanur and Kannan 2008], where each subcell has information about the optimal bundles of each agent. Next we check if this subcell also satisfy market clearing conditions, for which we generalize the [Devanur and Kannan 2008] construction, and finally we obtain a polynomial time query as in Theorem A.2 to check in each subcell.

Next we describe each step in detail:

Step 1 (*Optimal production schedule*). Recall that the optimal production schedule of firm f is given by the following optimization program:

$$
\begin{aligned}
\max \sum_{j \in \mathcal{P}^f} p_j x_j^{f,p} + \sum_{j \in \mathcal{R}^f} p_j x_j^{f,r} \\
\sum_{j \in \mathcal{P}^f} D_{jk}^f x_j^{f,p} \leq - \sum_{j \in \mathcal{R}^f} C_{jk}^f x_j^{f,r} + T_k^f, \ \forall k \\
x_j^{f,p} \geq 0, x_j^{f,r} \leq 0
\end{aligned}
\tag{3}
$$

Note that the feasible set is independent of prices, and the variables corresponding to the amount of raw goods are intentionally chosen to be non-positive (see Remark 2.2), which is crucial for the algorithm – it will later help us in capturing profit. Further without loss of generality we may assume that it is a full dimensional polytope, say $\mathcal{P}$, and no constraint is redundant. At any given prices, the set of optimal production schedules will be convex hull of a set of vertices of $\mathcal{P}$. We are interested in the reverse question: Given a vertex v of $\mathcal{P}$, at what prices v is an optimal production plan for firm f? To answer this, we partition the price space by the following set of hyperplanes for every two vertices v and v' of $\mathcal{P}$,

$$\boldsymbol{p}.v = \boldsymbol{p}.v'.$$

For each cell in this partition, one can identify the set of optimal production schedules, $\mathcal{Q}^f$ for each firm f, (i.e., convex hull of all the vertices which are optimal). From this, we can also deduce how much profit each firm makes in this particular cell as a linear function of prices. For an equilibrium in this cell, next we want to check if a price vector here also gives an optimal bundle to each agent so that market clears.

Step 2 (*Optimal bundle*). Recall the optimal bundle of agent i is given by the following optimization program:

$$
\begin{aligned}
&\max u_i \\
&u_i \leq \sum_j U_{jk}^i x_j^i + T_k^i, \ \forall k \\
\sum_j x_j^i p_j = &\sum_j W_j^i p_j + \sum_f \Theta_f^i \phi^f \\
&x_j^i \geq 0, \ \forall j \in \mathcal{G}
\end{aligned}
\tag{4}
$$

Here the budget constraint requires the profit obtained by each firm f. We substitute ϕ^f by the corresponding profit of the cell in Step 1, and make this constraint independent of the production. Now we use the construction of [Devanur and Kannan 2008] with the modified budget constraint and obtain the hyperplanes, Q^i for each agent i, to further partition this cell into subcells. For the sake of completeness, we give its details in Appendix B.

Now given a subcell, we have a polytope $\mathcal{Q}^f$ capturing optimal production schedules of each firm f, and a polytope $\hat{\mathcal{Q}}^i = \mathcal{Q}^i \cap \{\sum_j x_j^i p_j = \sum_j W_j^i p_j + \sum_f \Theta_f^i \phi^f\}$ capturing optimal bundles of each agent i. Next we need to check if $\exists(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p}) \in \mathcal{Q}^f, \forall f \in \mathcal{F}$ and $\exists \boldsymbol{x}^i \in \hat{\mathcal{Q}}^i, \forall i \in \mathcal{A}$ so that market clears.

Step 3 (*Market clearing*). There are two main difficulties here: $\hat{\mathcal{Q}}^i$ depends on prices and the total number of variables (search space) is $mn+nl$, which is very large. Both of these are also faced by [Devanur and Kannan 2008] and we extend their construction to handle these. For the second, we use LP duality to reduce the dimension of search space.

LEMMA 3.1 ([DEVANUR AND KANNAN 2008], EXTENDED). *Given polytopes $\mathcal{Q}^f$ for each firm f and $\hat{\mathcal{Q}}^i$ for each agent i, $\exists (\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p}) \in \mathcal{Q}^f, \boldsymbol{x}^i \in \hat{\mathcal{Q}}^i$ such that $\sum_i x^i_j - \sum_f x^{f,r}_j = 1 + \sum_f x^{f,p}_j, \forall j \in \mathcal{G}$ if and only if $\forall \boldsymbol{q} \in \mathcal{R}^n, \sum_j q_j \leq \sum_i \max_{\boldsymbol{x}^i \in \hat{\mathcal{Q}}^i} \boldsymbol{q}.\boldsymbol{x}^i - \sum_f \min_{(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p}) \in \mathcal{Q}^f} (\boldsymbol{q}.\boldsymbol{x}^{f,r} + \boldsymbol{q}.\boldsymbol{x}^{f,p})$.*

PROOF. Suppose there exists $\breve{\boldsymbol{x}}^i \in \hat{\mathcal{Q}}^i$ and $(\breve{\boldsymbol{x}}^{f,r}, \breve{\boldsymbol{x}}^{f,p}) \in \mathcal{Q}^f$ such that market clears, i.e., $\sum_i \breve{x}^i_j - \sum_f \breve{x}^{f,r}_j = 1 + \sum_f \breve{x}^{f,p}_j, \forall j \in \mathcal{G}$ then we have

$$\boldsymbol{q}.\boldsymbol{1} = \boldsymbol{q}.(\sum_i \breve{\boldsymbol{x}}^i - \sum_f (\breve{\boldsymbol{x}}^{f,r} + \breve{\boldsymbol{x}}^{f,p})) \leq \sum_i \max_{\boldsymbol{x}^i \in \hat{\mathcal{Q}}^i} \boldsymbol{q}.\boldsymbol{x}^i - \sum_f \min_{(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p}) \in \mathcal{Q}^f} (\boldsymbol{q}.\boldsymbol{x}^{f,r} + \boldsymbol{q}.\boldsymbol{x}^{f,p})$$

For the other direction, consider the set $\mathcal{P} = \{\boldsymbol{y} \in \mathcal{R}^n \mid \boldsymbol{y} = \sum_i \boldsymbol{x}^i - \sum_f (\boldsymbol{x}^{f,r} + \boldsymbol{x}^{f,p}),\ \boldsymbol{x}^i \in \hat{\mathcal{Q}}^i, (\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p}) \in \mathcal{Q}^f\}$. Using LP duality, if $\boldsymbol{1} \notin \mathcal{P}$ implies that there exists $\boldsymbol{q} \in \mathcal{R}^n$ and $w \in \mathcal{R}$ such that $\boldsymbol{q}.\boldsymbol{1} > w$ and $\boldsymbol{q}.\boldsymbol{y} \leq w, \forall \boldsymbol{y} \in \mathcal{P}$. Using this, we get

$$\boldsymbol{q}.\boldsymbol{1} > \max_{\boldsymbol{y} \in \mathcal{P}} \boldsymbol{q}.\boldsymbol{y} = \sum_i \max_{\boldsymbol{x}^i \in \hat{\mathcal{Q}}^i} \boldsymbol{q}.\boldsymbol{x}^i - \sum_f \min_{(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p}) \in \mathcal{Q}^f} (\boldsymbol{q}.\boldsymbol{x}^{f,r} + \boldsymbol{q}.\boldsymbol{x}^{f,p}).$$

□

Next we use the above lemma to capture market clearing. Essentially, we will partition the $(\boldsymbol{p}, \boldsymbol{q})$ space ($\mathcal{R}^{2n}$) by a set of hyperplanes so that for each cell in the partition, we can obtain both $\min_{(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p}) \in \mathcal{Q}^f} (\boldsymbol{q}.\boldsymbol{x}^{f,r} + \boldsymbol{q}.\boldsymbol{x}^{f,p}), \forall f$ and $\max_{\boldsymbol{x}^i \in \hat{\mathcal{Q}}^i} \boldsymbol{q}.\boldsymbol{x}^i, \forall i$ at some particular vertices, say v^i_* and v^f_*, of the respective polytopes and that depends only on the cell.

Recall that $\mathcal{Q}^f$ is independent of price variables and we partition the $(\boldsymbol{p}, \boldsymbol{q})$ space ($\mathcal{R}^{2n}$) by the following set of hyperplanes for every two vertices v^f and $\hat{v}^f$ of $\mathcal{Q}^f$:

$$\boldsymbol{q}.v^f = \boldsymbol{q}.\hat{v}^f.$$

Now consider $\hat{\mathcal{Q}}^i = \mathcal{Q}^i \cap \{\sum_j x^i_j p_j = \sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f\}$, where Q^i is independent of the prices. A vertex of $\hat{\mathcal{Q}}^i$ is a solution of n linearly independent equations and one of them is $\sum_j x^i_j p_j = \sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f$, so each coordinate of a vertex of $\hat{\mathcal{Q}}^i$ may be written as a ratio, whose numerator is independent of price variables $\boldsymbol{p}$ and denominator is a linear function of $\boldsymbol{p}$. Next partition the $(\boldsymbol{p}, \boldsymbol{q})$ space by the following set of hyperplanes for every two vertices v^i and $\hat{v}^i$ of $\hat{\mathcal{Q}}^i$:

$$\boldsymbol{q}.v^i = \boldsymbol{q}.\hat{v}^i.$$

Note that each of these equations is a polynomial of degree $n+1$ in variables $(\boldsymbol{p}, \boldsymbol{q})$. Given a cell in this partition, we can easily determine the vertices v^i_* and v^f_* taking the maximum and minimum value respectively.

Step 4 (*Final query*). For each cell, we need to solve the following problem:

$$\exists ? \boldsymbol{p} :\ \forall \boldsymbol{q},\ \sum_j q_j \leq \sum_i \boldsymbol{q}.v^i_* - \sum_f \boldsymbol{q}.v^f_*.$$

From the above analysis, it is clear that this is a polynomial inequality after clearing the denominator and Theorem A.2 is directly applicable.

4. PRODUCTION TO EXCHANGE

In this section, we give a general reduction from an Arrow-Debreu (AD) market (including production) $\mathcal{M}$ to an exchange market $\tilde{\mathcal{M}}$. Recall that $\mathcal{M}$ consists of a set of agents, a set of goods, and a set of firms. The idea is to replace each firm in $\mathcal{M}$ by a new agent in $\tilde{\mathcal{M}}$ which exactly behaves like the corresponding firm.

Recall that $\mathcal{R}^f$ and $\mathcal{P}^f$ respectively denote the set of raw and produced goods for firm f, and these two sets are disjoint. For simplicity, first we describe the reduction when production capabilities of each firm f is given by a function $F^f : \mathbb{R}_+^{|\mathcal{R}^f|} \rightarrow \mathbb{R}_+^{|\mathcal{P}^f|}$, where $F^f(\boldsymbol{x}^{f,r}) = \{F_j^f(\boldsymbol{x}^{f,r}) : j \in \mathcal{P}^f\}$ for a bundle $\boldsymbol{x}^{f,r}$ of raw goods. Let $\boldsymbol{x}^{f,p}$ denote the bundle of produced good, where $x_j^{f,p} = F_j^f(\boldsymbol{x}^{f,r}), \forall j \in \mathcal{P}^f$. Let Δ denote the maximum quantity of any good that can be produced in $\mathcal{M}$. Since total initial endowment of each good is finite and no production out of nothing, Δ is finite[2].

Given an AD market $\mathcal{M}$, construct $\tilde{\mathcal{M}}$ as in Table I. Consider the one-to-one mapping between equilibria of $\mathcal{M}$ and $\tilde{\mathcal{M}}$ as in Table II, where ϕ^f captures the profit of firm f.

Table I. Reduction from an AD market $\mathcal{M}$ to an exchange market $\tilde{\mathcal{M}}$

$$\mathcal{M} = (\mathcal{A}, \mathcal{G}, \mathcal{F}) \text{ with } (W, U, \Theta, F),\ m \stackrel{\text{def}}{=} |\mathcal{A}|,\ n \stackrel{\text{def}}{=} |\mathcal{G}|,\ l \stackrel{\text{def}}{=} |\mathcal{F}|$$

(Each agent is indexed by $i \in [m]$, good by $j \in [n]$, and firm by $f \in [l]$)

W_j^i : Endowment of good j with agent i

Θ_f^i : Profit share of agent i in firm f

$u_i : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$: Utility function of agent i

$F^f(\boldsymbol{x}^r) : \mathbb{R}_+^{|\mathcal{R}^f|} \rightarrow \mathbb{R}_+^{|\mathcal{P}^f|}$: Production function of firm f

$$\tilde{\mathcal{M}} = (\tilde{\mathcal{A}}, \tilde{\mathcal{G}}) \quad \text{with } (\tilde{W}, \tilde{U}),\ |\tilde{\mathcal{A}}| = m + l,\ |\tilde{\mathcal{G}}| = n + l$$

$$\tilde{\mathcal{A}} = \{i \mid i \in \mathcal{A}\} \cup \{m + f \mid f \in \mathcal{F}\}$$

$$\tilde{\mathcal{G}} = \{j \mid j \in \mathcal{G}\} \cup \{n + f \mid f \in \mathcal{F}\}$$

$$\tilde{W}_j^i = \begin{cases} \begin{cases} W_j^i & \text{if } j \le n \\ \Theta_f^i & \text{if } j = n + f \end{cases} & \text{if } i \le m \\[2ex] \begin{cases} \Delta & \text{if } j \in \mathcal{P}^f \\ 0 & \text{Otherwise} \end{cases} & \text{if } i > m \end{cases}$$

$\tilde{u}_i : \mathbb{R}_+^{n+l} \rightarrow \mathbb{R}_+$: Utility function of agent i in $\tilde{\mathcal{M}}$

$\tilde{\boldsymbol{x}}^i \in \mathbb{R}_+^{n+l}$, $\tilde{\boldsymbol{x}}^i_{|_\mathcal{G}} = \{\tilde{x}_j^i : j \in \mathcal{G}\}$ and $\tilde{\boldsymbol{x}}^i_{|_{\mathcal{R}^f}} = \{\tilde{x}_j^i : j \in \mathcal{R}^f\}$

$$\tilde{u}_i(\tilde{\boldsymbol{x}}^i) = \begin{cases} u_i(\tilde{\boldsymbol{x}}^i_{|_\mathcal{G}}) & \text{if } i \le m \\[2ex] \min \left\{ \begin{array}{ll} F_j^f(\tilde{\boldsymbol{x}}^i_{|_{\mathcal{R}^f}}) + \tilde{x}_j^i, & j \in \mathcal{P}^f \\ \Delta \tilde{x}_{n+f}^i & \end{array} \right\} & \text{if } i = m + f \end{cases}$$

[2] In case of PLC markets, bit length of Δ is polynomial in the size of input and can be computed in polynomial time [Garg et al. 2014a].

Table II. One-to-one mapping between equilibria of $\mathcal{M}$ and $\tilde{\mathcal{M}}$

$\mathcal{M} \longleftrightarrow \tilde{\mathcal{M}}$ $(\boldsymbol{x},\boldsymbol{p}) \longleftrightarrow (\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}})$	
$(\boldsymbol{x},\boldsymbol{p}) \longrightarrow (\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}})$	$(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}}) \longrightarrow (\boldsymbol{x},\boldsymbol{p})$
$\tilde{x}^i_j = \begin{cases} \begin{cases} x^i_j & \text{if } j \le n \\ 0 & \text{if } j > n \end{cases} & \text{if } i \le m \\ \begin{cases} x^{f,r}_j & \text{if } j \in \mathcal{R}^f \\ \Delta - x^{f,p}_j & \text{if } j \in \mathcal{P}^f \\ 1 & \text{if } j = n+f \end{cases} & \text{if } i = m+f \end{cases}$ $\phi^f = \sum_{j\in\mathcal{P}^f} x^{f,p}_j p_j - \sum_{j\in\mathcal{R}^f} x^{f,r}_j p_j$ $\tilde{p}_j = \begin{cases} p_j & \text{if } j \le n \\ \phi^f & \text{if } j = n+f \end{cases}$	$x^i_j = \tilde{x}^i_j$ $x^{f,r}_j = \tilde{x}^{m+f}_j$ $x^{f,p}_j = \Delta - \tilde{x}^{m+f}_j$ $p_j = \tilde{p}_j$

LEMMA 4.1. *If $(\boldsymbol{x},\boldsymbol{p})$ is an equilibrium for $\mathcal{M}$, then $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}})$, as per Table II, is an equilibrium for $\tilde{\mathcal{M}}$.*

PROOF. We need to show that $\tilde{\boldsymbol{x}}$ gives an optimal bundle to each agent in $\tilde{\mathcal{M}}$ at prices $\tilde{\boldsymbol{p}}$ and market clears. For this, first consider an agent $i \le m$ of $\tilde{\mathcal{M}}$, whose budget at prices $\tilde{\boldsymbol{p}}$ is

$$\sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f.$$

This is same as the budget of agent i in $\mathcal{M}$ at prices $\boldsymbol{p}$. Since utility function of agent $i \le m$ of $\tilde{\mathcal{M}}$ is same as the utility function of agent i in $\mathcal{M}$, i.e., no utility from goods $j > n$, $\tilde{\boldsymbol{x}}$ gives an optimal bundle to agent i at prices $\tilde{\boldsymbol{p}}$.

Next consider an agent $i = m+f$ of $\tilde{\mathcal{M}}$. At $\tilde{\boldsymbol{x}}$, it gets Δ units of utility. Suppose there is another bundle, say $\hat{\boldsymbol{x}}^i$, which gives more utility than Δ, then we have

$$\begin{aligned} F^f_j(\hat{\boldsymbol{x}}^i|_{\mathcal{R}^f}) + \hat{x}^i_j &> \Delta, \qquad \forall j \in \mathcal{P}^f \\ \hat{x}^i_{n+f} &> 1 \end{aligned} \tag{5}$$

Further, $\hat{\boldsymbol{x}}^i$ must also satisfy budget constraint, which is

$$\sum_{j\in\mathcal{R}^f} \hat{x}^i_j \tilde{p}_j + \sum_{j\in\mathcal{P}^f} \hat{x}^i_j \tilde{p}_j + \hat{x}^i_{n+f}\tilde{p}_{n+f} \;\le\; \Delta \sum_{j\in\mathcal{P}^f} \tilde{p}_j$$

Using (5) and substituting $\tilde{\boldsymbol{p}}$ from Table II, we have

$$\phi^f \;\le\; \hat{x}^i_{n+f}\phi^f \;\le\; \sum_{j\in\mathcal{P}^f} (\Delta - \hat{x}^i_j)p_j - \sum_{j\in\mathcal{R}^f} \hat{x}^i_j p_j \;<\; \sum_{j\in\mathcal{P}^f} F^f_j(\hat{\boldsymbol{x}}^i|_{\mathcal{R}^f})p_j - \sum_{j\in\mathcal{R}^f} \hat{x}^i_j p_j$$

This implies that firm f in $\mathcal{M}$ can earn more profit using the bundle $\hat{\boldsymbol{x}}^i$ rather than $\boldsymbol{x}^f$ at prices $\boldsymbol{p}$, which is a contradiction. Further, it is easy to verify that each agent satisfies the budget constraint at $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$. Therefore, $\tilde{\boldsymbol{x}}$ gives an optimal bundle to each agent at prices $\tilde{\boldsymbol{p}}$. For market clearing, we need to show that

$$\forall j: \sum_{i=1}^{m+l} \tilde{x}_j^i = \sum_{i=1}^{m+l} W_j^i.$$

For $j \leq n$, it is

$$\sum_{i=1}^{m} x_j^i + \sum_{\forall f: j \in \mathcal{R}^f} x_j^{f,r} + \sum_{\forall f: j \in \mathcal{P}^f} \Delta - \sum_{\forall f: j \in \mathcal{P}^f} x_j^{f,p} = \sum_{i=1}^{m} W_j^i + \sum_{\forall f: j \in \mathcal{P}^f} \Delta$$

For $j > n$, it is

$$1 = \sum_{i=1}^{m} \Theta_{j-n}^i,$$

which follows from the market clearing at $\boldsymbol{x}$ in $\mathcal{M}$. $\square$

LEMMA 4.2. *If $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ is an equilibrium for $\tilde{\mathcal{M}}$, then $(\boldsymbol{x}, \boldsymbol{p})$, as per Table II, is an equilibrium for $\mathcal{M}$.*

PROOF. At $\boldsymbol{p}$, $\boldsymbol{x}^i = \tilde{\boldsymbol{x}}^i$ is an optimal bundle for agent i because utility function $u_i(.) = \tilde{u}_i(.)$ for $i \leq m$, $p_j = \tilde{p}_j$ for $j \leq n$ and the budget $\sum_j W_j^i p_j + \sum_f \Theta_f^i \phi^f$ is same in both $\mathcal{M}$ and $\tilde{\mathcal{M}}$. Next we show that $(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p})$ gives an optimal production schedule for firm f at prices $\boldsymbol{p}$. Suppose there is another schedule $(\hat{\boldsymbol{x}}^{f,r}, \hat{\boldsymbol{x}}^{f,p} = F^f(\hat{\boldsymbol{x}}^{f,r}))$ at prices $\boldsymbol{p}$, which is more profitable than $(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p})$. Then, we have

$$\sum_{j \in \mathcal{P}^f} x_j^{f,p} p_j - \sum_{j \in \mathcal{R}^f} x_j^{f,r} p_j < \sum_{j \in \mathcal{P}^f} \hat{x}_j^{f,p} p_j - \sum_{j \in \mathcal{R}^f} \hat{x}_j^{f,r} p_j$$

Using Table II, we get

$$\sum_{j \in \mathcal{P}^f} (\Delta - \tilde{x}_j^{m+f}) p_j - \sum_{j \in \mathcal{R}^f} \tilde{x}_j^{m+f} p_j < \sum_{j \in \mathcal{P}^f} \hat{x}_j^{f,p} p_j - \sum_{j \in \mathcal{R}^f} \hat{x}_j^{f,r} p_j$$

$$\sum_{j \in \mathcal{P}^f} (\Delta - \hat{x}_j^{f,p}) p_j + \sum_{j \in \mathcal{R}^f} \hat{x}_j^{f,r} p_j < \sum_{j \in \mathcal{P}^f} \tilde{x}_j^{m+f} p_j + \sum_{j \in \mathcal{R}^f} \tilde{x}_j^{m+f} p_j,$$

which implies that the bundle $(\hat{x}_j^{f,r}, \forall j \in \mathcal{R}^f; (\Delta - \hat{x}_j^{f,p}), \forall j \in \mathcal{P}^f)$ and 1 unit of good $m+f$ also gives Δ units of utility to agent $m+f$ at prices $\tilde{\boldsymbol{p}}$, and it is cheaper than the bundle $\tilde{\boldsymbol{x}}^{m+f}$. Therefore, agent $m+f$ can earn more than Δ units of utility at prices $\tilde{\boldsymbol{p}}$, which is a contradiction since $\tilde{\boldsymbol{x}}^{m+f}$ is an optimal bundle at prices $\tilde{\boldsymbol{p}}$. Further, market clearing in $\mathcal{M}$ easily follows from the market clearing in $\tilde{\mathcal{M}}$. $\square$

The next theorem easily follows from the construction.

THEOREM 4.3. *The reduction of Table I reduces an Arrow-Debreu market with nested CES-Leontief utilities and nested CES-Leontief production to an equivalent exchange market with nested CES-Leontief utilities.*

4.1. Production: A correspondence

In general, production capabilities of a firm f is given by a correspondence $F^f : \mathbb{R}_+^{|\mathcal{R}^f|} \rightrightarrows \mathbb{R}_+^{|P^f|}$. In this section, we extend the reduction of Section 4 to this case. Earlier, when it was a function then there was a unique bundle of produced goods, that can be produced from a given bundle of raw goods. Now it is a multiset and hence there is a choice. In this case, from a given bundle of raw goods, the firm chooses a bundle, among all choices, which maximizes its profit. Let $x^{f,r}$ denote a bundle of raw goods, and let $\boldsymbol{x}^{f,p} \in F^f(\boldsymbol{x}^{f,r})$ denote the bundle of produced good, which maximizes f's profit.

The reduction of Table I is modified in Table III to handle this case. Note that only the utility function of agents in $\tilde{\mathcal{M}}$ has been modified. Also observe that when the correspondence F^f is a function then it exactly matches with Table I.

Table III. Reduction from AD market to exchange market

$$
\begin{array}{ll}
\mathcal{M} = & (\mathcal{A},\mathcal{G},\mathcal{F}) \text{ with } (W,U,\Theta,F),\ m=|\mathcal{A}|, n=|\mathcal{G}|, l=|\mathcal{F}| \\
& \text{(Each agent is indexed by } i\in[m]\text{, good by } j\in[n]\text{, and firm by } f\in[l]\text{)} \\
& W_j^i : \text{Endowment of good } j \text{ with agent } i \\
& \Theta_f^i : \text{Profit share of agent } i \text{ in firm } f \\
& u_i : \mathbb{R}_+^n \to \mathbb{R}_+ : \text{Utility function of agent } i \\
& F^f(\boldsymbol{x}^r) : \mathbb{R}_+^{|\mathcal{R}^f|} \rightrightarrows \mathbb{R}_+^{|\mathcal{P}^f|} : \text{Production correspondence of firm } f
\end{array}
$$

$$
\begin{array}{lll}
\tilde{\mathcal{M}} = & (\tilde{\mathcal{A}},\tilde{\mathcal{G}}) & \text{with } (\tilde{W},\tilde{U}),\ |\tilde{\mathcal{A}}| = m+l, |\tilde{\mathcal{G}}| = n+l \\
& \tilde{\mathcal{G}} = & \{j \mid j\in\mathcal{G}\} \cup \{n+f \mid f\in\mathcal{F}\} \\
& \tilde{\mathcal{A}} = & \{i \mid i\in\mathcal{A}\} \cup \{m+f \mid f\in\mathcal{F}\}
\end{array}
$$

$$
\tilde{W}_j^i = \begin{cases} \begin{cases} W_j^i & \text{if } j\le n \\ \Theta_f^i & \text{if } j = n+f \end{cases} & \text{if } i\le m \\ \\ \begin{cases} \Delta & \text{if } j\in\mathcal{P}^f \\ 0 & \text{Otherwise} \end{cases} & \text{if } i > m \end{cases}
$$

$$
\tilde{U}_i : \mathbb{R}_+^{n+l} \to \mathbb{R}_+ : \text{Utility function of agent } i \text{ in } \tilde{\mathcal{M}}
$$

$$
\tilde{\boldsymbol{x}}^i \in \mathbb{R}_+^{n+l},\ \ \tilde{\boldsymbol{x}}^i_{|_\mathcal{G}} = \{\tilde{x}_j^i : j\in\mathcal{G}\} \text{ and } \tilde{\boldsymbol{x}}^i_{|_{\mathcal{R}^f}} = \{\tilde{x}_j^i : j\in\mathcal{R}^f\}
$$

$$
\tilde{U}_i(\tilde{\boldsymbol{x}}^i) = \begin{cases} u_i(\tilde{\boldsymbol{x}}^i_{|_\mathcal{G}}) & \text{if } i\le m \\ \\ \min\left\{ \begin{array}{c} \max\limits_{\tilde{\boldsymbol{y}}^i \in F^f(\tilde{\boldsymbol{x}}^i_{|_{\mathcal{R}^f}})} \min\limits_{j\in\mathcal{P}^f} \{\tilde{y}_j^i + \tilde{x}_j^i\} \\ \Delta \tilde{x}_{n+f}^i \end{array} \right\} & \text{if } i = m+f \end{cases}
$$

The one-to-one mapping between equilibria of $\mathcal{M}$ and $\tilde{\mathcal{M}}$ remains same as in Table II. The proof of correctness is essentially same as Lemmas 4.1 and 4.2. Next we show that the utility function of newly created agents of $\tilde{\mathcal{M}}$, corresponding to firms of $\tilde{\mathcal{M}}$, in Table III is not arbitrary, but *concave*.

LEMMA 4.4. *The utility function of each agent $i > m$, defined by $\tilde{U}_i(\tilde{\boldsymbol{x}})$ in Table III, is concave when the set of production capabilities, defined by the correspondence F^f, is convex.*

PROOF. Let $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ be two bundles of goods in $\mathbb{R}_+^{n+l}$. We need to show that

$$u_i(\lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) \geq \lambda u_i(\boldsymbol{x}^1) + (1-\lambda)u_i(\boldsymbol{x}^2), \forall \lambda \in [0,1], \forall i > m.$$

Consider an agent $i = m+f$. Let $g(\boldsymbol{x}) = \max_{\boldsymbol{y} \in F^f(\boldsymbol{x}_{|_{\mathcal{R}^f}})} \min_{j \in \mathcal{P}^f} \{y_j + x_j\}$. Let $\boldsymbol{y}^k \in F^f(\boldsymbol{x}^k_{|_{\mathcal{R}^f}})$ for $k = 1,2$ be the bundle picked in $g(\boldsymbol{x}^1)$ and $g(\boldsymbol{x}^2)$ respectively, and let $g(\boldsymbol{x}^k) = y^k_{j_k} + x^k_{j_k}$ for $k = 1,2$. Since the production capabilities of firm f is a convex set, we have

$$\lambda \boldsymbol{y}^1 + (1-\lambda)\boldsymbol{y}^2 \in F^f(\lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) \tag{6}$$

Let $\boldsymbol{y}$ be the bundles picked in $g(\lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2)$ and $y_j + x_j = g(\lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2)$, where $x_j = \lambda x^1_j + (1-\lambda)x^2_j$. So we have

$$u_i(\boldsymbol{x}^k) = \min\{y^k_{j_k} + x^k_{j_k}, \Delta x^k_{n+f}\}, \text{ for } k = 1,2, \text{ and}$$

$$u_i(\lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) = \min\{y_j + x_j, \Delta(\lambda x^1_{n+f} + (1-\lambda)x^2_{n+f})\}.$$

Since $g(\boldsymbol{x})$ picks the minimum coordinate value, we have

$$y^k_j + x^k_j \geq y^k_{j_k} + x^k_{j_k}, \text{ for } k = 1,2 \tag{7}$$

Using (6) and (7), we get

$$\begin{aligned} y_j + x_j &\geq \lambda y^1_j + (1-\lambda)y^2_j + x_j \\ &= \lambda(y^1_j + x^1_j) + (1-\lambda)(y^2_j + x^2_j) \\ &\geq \lambda(y^1_{j_1} + x^1_{j_1}) + (1-\lambda)(y^2_{j_2} + x^2_{j_2}) \end{aligned}$$

This implies that

$$\begin{aligned} u_i(\lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) &\geq \min\{\lambda(y^1_{j_1} + x^1_{j_1}) + (1-\lambda)(y^2_{j_2} + x^2_{j_2}), \Delta(\lambda x^1_{n+f} + (1-\lambda)x^2_{n+f})\} \\ &\geq \lambda \min\{y^1_{j_1} + x^1_{j_1}, \Delta x^1_{n+f}\} + (1-\lambda)\min\{y^2_{j_2} + x^2_{j_2}, \Delta x^2_{n+f}\} \\ &= \lambda u_i(\boldsymbol{x}^1) + (1-\lambda)u_i(\boldsymbol{x}^2) \end{aligned}$$

□

THEOREM 4.5. *An Arrow-Debreu market $\mathcal{M}$ with piecewise-linear concave (PLC) utilities and polyhedral production set reduces to an exchange market $\tilde{\mathcal{M}}$ with PLC utilities.*

PROOF. We need to show that utility function of each agent $i = m+f$, corresponding to firm f, in $\tilde{\mathcal{M}}$ is PLC. Let $u_i(.)$ be the utility function of agent i. It is easy to check that $u_i(\boldsymbol{x}^i)$, defined in Table III, is the solution of following linear program (LP):

$$\begin{aligned} &\max u_i \\ u_i \;&\leq\; \Delta x^i_{n+f} \\ u_i \;&\leq\; x^i_j + y^i_j, \; \forall j \in \mathcal{P}^f \\ \sum_{j \in \mathcal{P}^f} D^f_{jk} y^i_j \;&\leq\; \sum_{j \in \mathcal{R}^f} C^f_{jk} x^i_j + T^f_k, \; \forall k \\ x^i_j &\geq 0, \; \forall j \\ y^i_j &\geq 0, \; \forall j \in \mathcal{P}^f \end{aligned} \tag{8}$$

Note that the feasible region of (8) is a polyhedral set in $(u_i, \boldsymbol{x}^i, \boldsymbol{y}^i)$. The projection of it onto $(u_i, \boldsymbol{x}^i)$ is again a polyhedral set and its boundary gives the value of $u_i(\boldsymbol{x}^i)$ for the associated $\boldsymbol{x}^i$. Therefore, it is a PLC function. $\square$

4.2. Rader's reduction

Rader [Rader 1964] considered the following market setting: Agent i has W^i_j amount of good j as an initial endowment and a production capabilities set Y^i. Each agent i has a utility function u^i over consumption of goods. Let $x^i_{i'j}$ is the amount of good j, agent i trades with agent i', where $x^i_{i'j} < 0$ means good is received by i and $x^i_{i'j} > 0$ means good is received by i'. When there is no production, then the net consumption of i is $\boldsymbol{z}^i = \boldsymbol{W}^i + \sum_{i'} \boldsymbol{x}^i_{i'}$, where $\boldsymbol{z}^i$ is required to be non-negative.

In case of production, let $\boldsymbol{y}^i \in Y^i$, where negative coordinate means raw good and positive coordinate means produced good. He proposed the following induced utility function u^i_*:

$$u^i_*(\boldsymbol{z}^i) = \max_{\boldsymbol{y}^i \in Y^i \ s.t.\ \boldsymbol{z}^i + \boldsymbol{y}^i \geq 0} u^i(\boldsymbol{z}^i + \boldsymbol{y}^i).$$

This is a nice reduction (also called *Principle of Equivalence*) which preserves concavity and monotonicity of utility function given that the production capabilities set is convex. However as Rader said, these utility functions are "*of course artificially constructed*" since they are not in any standard form, and moreover they are defined on variables which can take negative values and are not even related to the amount of goods agents consume in the market. Further we note that market clearing conditions and budget constraints are different in this market setting.

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REFERENCES

K. Arrow and G. Debreu. 1954. Existence of an Equilibrium for a Competitive Economy. *Econometrica* 22 (1954), 265–290.

S. Basu, R. Pollack, and M.-F. Roy. 1996. On the combinatorial and algebraic complexity of quantifier elimination. *JACM* 43(6) (1996), 1002–1045.

S. Basu, R. Pollack, and M.-F. Roy. 2004. *Quantifier Elimination and Cylindrical Algebraic Decomposition*. Springer.

W. C. Brainard and H. E. Scarf. 2000. How to Compute Equilibrium Prices in 1891. *Cowles Foundation Discussion Paper* 1270 (2000).

X. Chen, D. Dai, Y. Du, and S.-H. Teng. 2009. Settling the complexity of Arrow-Debreu equilibria in markets with additively separable utilities. In *FOCS*. 273–282.

X. Chen and S.-H. Teng. 2009. Spending is not easier than trading: on the computational equivalence of Fisher and Arrow-Debreu equilibria. In *ISAAC*. 647–656.

Bruno Codenotti, Sriram Pemmaraju, and Kasturi Varadarajan. 2005. On the polynomial time computation of equilibria for certain exchange economies. In *SODA*. 72–81.

O. de La Grandville. 2009. *Economic growth: A unified approach*. Cambridge University Press.

N. Devanur and R. Kannan. 2008. Market Equilibria in Polynomial Time for Fixed Number of Goods or Agents. In *FOCS*. 45–53.

N. Devanur, C. H. Papadimitriou, A. Saberi, and V. V. Vazirani. 2008. Market Equilibrium via a Primal-Dual Algorithm for a Convex Program. *JACM* 55, 5 (2008).

Ran Duan and Kurt Mehlhorn. 2013. A Combinatorial Polynomial Algorithm for the Linear Arrow-Debreu Market. In *ICALP*. 425–436.

Jugal Garg, Ruta Mehta, and Vijay V. Vazirani. 2014a. Dichotomies in Equilibrium Computation, and Complementary Pivot Algorithms for a New Class of Non-Separable Utility Functions. In *STOC*. 525–534.

Jugal Garg, Ruta Mehta, Vijay V. Vazirani, and Sadra Yazdanbod. 2014b. Leontief Exchange Markets Can Solve Multivariate Polynomial Equations, Yielding FIXP and ETR Hardness. In *arXiv:1411.5060*.

Jugal Garg and Vijay V. Vazirani. 2014. On computability of Equilibria in Markets with Production. In *SODA*. 1329–1340.

K. Jain. 2007. A polynomial time algorithm for computing the Arrow-Debreu market equilibrium for linear utilities. *SIAM J. Comput.* 37, 1 (2007), 306–318.

K. Jain and K. Varadarajan. 2006. Equilibria for economies with production: Constant-returns technologies and production planning constraints. In *SODA*. 688–697.

A. Mas-Colell. 1991. On the Uniqueness of Equilibrium Once Again. (1991).

J. B. Orlin. 2010. Improved algorithms for computing Fisher's market clearing prices. In *STOC*. 291–300.

T. Rader. 1964. Edgeworth Exchange and General Economic Equilibrium. *Yale Economic Essays* 4 (1964), 133–181.

T. Rutherford. 1999. Applied general equilibrium modeling with MPSGE as a GAMS subsystem: An overview of the modeling framework and syntax. *Computational Economics* 14 (1999), 1–46.

H. Scarf. 1967. The approximation of fixed points of a continuous mapping. *SIAM Journal of Applied Mathematics* 15, 1 (1967), 1328–1343.

J. B. Shoven and J. Whalley. 1992. *Applying general equilibrium*. Cambridge University Press.

S. Smale. 1976. A convergent process of price adjustment and global newton methods. *Journal of Mathematical Economics* 3(2) (1976), 107–120.

H. Sonnenschein. 1973. Do Walras' identity and continuity characterize the class of community excess-demand functions? *Journal of Economic Theory* 6 (1973), 345–354".

Vijay V. Vazirani and Mihalis Yannakakis. 2011. Market equilibrium under separable, piecewise-linear, concave utilities. *Journal of ACM* 58, 3 (2011), 10:1–10:25.

Laszlo A. Vegh. 2012. Strongly polynomial algorithm for a class of minimum-cost flow problems with separable convex objectives. In *STOC*.

Yinyu Ye. 2008. A path to the Arrow-Debreu competitive market equilibrium. *Math. Program.* 111(1-2) (2008), 315–348.

APPENDIX

A. RESULTS FROM COMPUTATIONAL ALGEBRAIC GEOMETRY

A set of degree d polynomials $q_1, \ldots, q_M \in \mathcal{R}[x_1, \ldots, x_n]$ partition the space into cells, where each cell is defined by the sign assignment $\sigma \in \{0, 1, -1\}^M$ to the polynomials. A basic fact about these cells is:

THEOREM A.1 ([BASU ET AL. 2004]). *The number of non-empty cells and the time required to enumerate them is* $O(M^{n+1})d^{O(n)}$.

Further, given a particular sign assignment $\sigma \in \{0, 1, -1\}^M$, one can check if there is an $\boldsymbol{x} \in \mathcal{R}^n$ such that $sign(q_1(\boldsymbol{x}), \ldots, q_M(\boldsymbol{x})) = \sigma$ (and output if there is one) by enumerating over all the non-empty cells. Moreover, when the total number of variables are constant, a more general problem, which has an existential and a universal quantifier, can be solved as in the following theorem:

THEOREM A.2 ([BASU ET AL. 1996]). *Given a set of degree d polynomials $q_1, \ldots, q_M \in \mathcal{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and a sign assignment $\sigma \in \{0, 1, -1\}^M$, the time required to check if there exists an $\boldsymbol{x} \in \mathcal{R}^n$ such that $\forall \boldsymbol{y} \in \mathcal{R}^n, sign(q_1(\boldsymbol{x}, \boldsymbol{y}), \ldots, q_M(\boldsymbol{x}, \boldsymbol{y})) = \sigma$, and output one if exists, is $O(M^{(n+1)^2})d^{O(n^2)}$.*

B. PROCEDURE FOR OPTIMAL BUNDLES

In this section, we summarize the procedure for optimal bundles given in [Devanur and Kannan 2008]. Recall that the optimal bundle of agent i is given by the following program:

$$
\begin{aligned}
&\max u_i \\
u_i &\leq \sum_j U^i_{jk} x^i_j + T^i_k,\ \forall k \in [K] \\
\sum_j x^i_j p_j &\leq \sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f \\
x^i_j &\geq 0,\ \forall j \in \mathcal{G}
\end{aligned}
\tag{9}
$$

Note that the budget constraint here is dependent on the profit earned by the firms, however we can treat it as a linear function of prices as explained in Step 2 of the algorithm in Section 3. Let $e^i \stackrel{\text{def}}{=} \sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f$. Further, without loss of generality we may assume that the constraints form a full-dimensional polytope and no constraint is redundant.

Next for some choices of $K^* \subseteq [K]$ and $J^* \subseteq [n]$ such that $|K^*| + |J^*| \leq n$, the set of optimal bundles is $\mathcal{Q}^i \cap \{\boldsymbol{x} : \boldsymbol{p}.\boldsymbol{x} = e^i\}$ where

$$
\begin{aligned}
\mathcal{Q}^i = \{\boldsymbol{x} : \quad &\forall k \in K^*,\ u_i = \boldsymbol{U}^i_k.\boldsymbol{x}^i + T^i_k; \quad \forall k \notin K^*,\ u_i \leq \boldsymbol{U}^i_k.\boldsymbol{x}^i + T^i_k, \\
&\forall j \in J^*,\ x^i_j = 0; \qquad\qquad \forall j \notin J^*,\ x^i_j \geq 0\}
\end{aligned}
$$

There are $O((K+n)^n)$ different choices for $\mathcal{Q}^i$ and the correct one depends on the prices. For this, we partition the price space so that the choice depends on the cell in this partition. We are actually interested in the reverse question: For what prices, (K^*, J^*) is the correct choice?

Next consider the dual problem of (9):

$$
\begin{aligned}
\text{minimize} \quad & \alpha^i e^i + \sum_k \lambda^i_k T^i_k \\
\text{subject to} \quad & \forall j,\ p_j \alpha^i \geq \sum_k \lambda^i_k U^i_{jk}; \quad \sum_k \lambda^i_k \geq 1 \\
& \alpha^i \geq 0,\ \lambda^i_k \geq 0
\end{aligned}
$$

The set of optimal solutions of dual is given by:

$$
\begin{aligned}
\mathcal{D}^i = \{\alpha^i, \lambda^i_k : \quad &\forall j \in J^{**}, p_j \alpha^i = \sum_k \lambda^i_k U^i_{jk}; \qquad \forall j \notin J^{**}, p_j \alpha^i \geq \sum_k \lambda^i_k U^i_{jk} \\
&\forall k \in K^{**}, \lambda^i_k = 0; \qquad \forall k \notin K^{**}, \lambda^i_k \geq 0; \qquad \sum_k \lambda^i_k = 1\}
\end{aligned}
$$

for some choices of $K^{**} \subseteq [K]$ and $J^{**} \subseteq [n]$.

Using complementarity slackness condition for the optimality, it is clear that if $K^{**} = (K^*)^c$ and $J^{**} = (J^*)^c$ then a given (K^*, J^*) is the right choice for those prices, for which both $\hat{\mathcal{Q}}^i$ and $\mathcal{D}^i$ is non-empty.

Next we want to partition the price space so that given a cell in this partition, one can determine if both $\hat{\mathcal{Q}}^i$ and $\mathcal{D}^i$ is non-empty. Note that $\mathcal{Q}^i$ is independent of prices, so clearly $\hat{\mathcal{Q}}^i$ is non-empty iff the hyperplane $\boldsymbol{p}.\boldsymbol{x} = e^i$ intersects $\mathcal{Q}^i$. Further this happens iff there are two vertices z and z' of $\mathcal{Q}^i$ such that $\boldsymbol{p}.z \leq e^i$ and $\boldsymbol{p}.z' \geq e^i$.

Note that the vertices of $\mathcal{Q}^i$ are same as the vertices of the simplicial subdivision in the definition of agent i's utility function. Next partition the price space with the following hyperplane for each vertex of the simplicial subdivision z:

$$\boldsymbol{p}.z = e^i.$$

As discussed above, a cell in this partition can determine if $\hat{\mathcal{Q}}^i$ is non-empty. For the non-emptiness of $\mathcal{D}^i$, consider the equations:

$$\forall j \in J^{**},\ p_j\alpha_i = \sum_{k\in K^*} \lambda_k^i U_{jk}^i \quad \text{and} \quad \sum_{k\in K^*} \lambda_k^i = 1$$

We can eliminate α_i from these equations and get:

$$\forall j \in J^{**}, \quad p_j = \sum_{k\in K^*} \mu_k^i U_{jk}^i.$$

Note that this is an over-determined system of equations, since the number of variables are $|K^*| \leq n - |J^*| = |J^{**}| =$ the number of equations. Hence one can solve for the μ_k^i's in terms of $\boldsymbol{p}$. In fact, each μ_k^i is a linear function of $\boldsymbol{p}$. In case the number of variables are strictly less than number of equations, then one can solve for μ_k^i's using a subset of equations and then plug in those values in the rest of equations, which again give a linear equations in $\boldsymbol{p}$. Finally, $\mathcal{D}^i$ is non-empty iff if the solution also satisfy:

$$\forall k \in K^*,\ \mu_k^i \geq 0 \quad \text{and} \quad \forall j \in J^*,\ p_j \geq \sum_{k\in K^*} \mu_k^i U_{jk}^i.$$

Note that all equations are linear in $\boldsymbol{p}$, so if we partition the space with these hyperplanes then a cell in the partition determines if $\mathcal{D}^i$ is non-empty. Moreover the total number of equations are polynomials and their degree are polynomial, and the number of variables are constant. Hence the running time is polynomial.